A MODAL LOGIC OF METRIC SPACES

M. SHEREMET, F. WOLTER, AND M. ZAKHARYASCHEV

In an attempt to extend Tarski's programme of algebraising topology to metric spaces, we introduce three binary operators on metric spaces (I, d): for $A, B \subseteq I$,

$$\begin{aligned} A \leftleftarrows B &= \{u \in I \mid d(u,A) < d(u,B)\}, \\ A \underline{\leftarrow} B &= \{u \in I \mid \forall b \in B \, \exists a \in A \, (d(u,a) < d(u,b))\}, \\ A \underline{\underline{\leftarrow}} B &= \{u \in I \mid \exists a \in A \, \forall b \in B \, (d(u,a) \leq d(u,b))\}, \end{aligned}$$

where $d(u,A) = \inf\{d(u,a) \mid a \in A\}$ if $A \neq \emptyset$, and $d(u,\emptyset) = +\infty$.

Denote by $\mathcal{L}$ the logic obtained by adding the operators $\leftleftarrows$, $\underline{\leftarrow}$, $\underline{\underline{\leftarrow}}$ to classical propositional logic and interpreting it over metric spaces (propositional variables are interpreted as their arbitrary subsets).

It is easy to see that $\mathcal{L}$ is more expressive than Tarski's $S4$; for example, $A \leftleftarrows \neg A$ is the interior of A. We show the following:

(1) $\mathcal{L}$ is as expressive over metric spaces as the logic $\mathcal{L}'$ with the operators $\exists x$ (there exists $x > 0$), $\exists^{<x}$ (in the open x-neighbourhood), and $\exists^{\leq x}$ (in the closed x-neighbourhood), where formulas starting with $\exists^{<x}$ or $\exists^{\leq x}$ are only allowed in a Boolean combination immediately after $\exists x$. For example, $A \leftleftarrows B$ is equivalent to $\exists x(\exists^{<x} A \sqcap \neg \exists^{<x} B)$.
(2) There is a natural Hilbert-style finite axiomatisation of $\mathcal{L}$.
(3) The decision problem for $\mathcal{L}$ is ExpTime-complete.
(4) Satisfiability of $\mathcal{L}$-formulas is undecidable over $\mathbb{R}$ (which is proved by reduction of the solvability problem for Diophantine equations).

SCHOOL OF COMPUTER SCIENCE AND INFORMATION SYSTEMS
BIRKBECK (UNIVERSITY OF LONDON)
MALET STREET, LONDON, WC1E 7HX, U.K.
E-mail: {mikhail,michael}@dcs.bbk.ac.uk

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF LIVERPOOL
LIVERPOOL, L69 3BX, U.K.
E-mail: {frank}@csc.liv.ac.uk

Meeting
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