Constructing Chains and Antichains in Partial Orders

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Definition

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A Turing ideal $\mathcal{I}$ is a model of **CAC** if whenever $\prec$ is a partial ordering which is computable (from some set) in $\mathcal{I}$, $\mathcal{I}$ contains either an infinite monotone sequence or an infinite antichain.

Certainly **CAC** implies **ADS** (i.e. every model of **CAC** is a model of **ADS**).

Definition

If $c : [\mathbb{N}]^2 \to C$ is a coloring of pairs, a color i is *transitive* if whenever $n < m < p$ and $c(n, m) = c(m, p) = i$ also $c(n, p) = i$.

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Theorem (Hirschfelt-Shore)

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2. *A Turing ideal $\mathcal{I}$ is a model of* **CAC** *iff whenever c is a 2-coloring of pairs where one color is transitive, $\mathcal{I}$ contains an infinite homogeneous set.*

Definition

If $\prec$ is a partial ordering, a *bad sequence* is a sequence $(x_1, x_2, \ldots)$ so that, for all $n < m$, $n \not\preceq m$. A *well-partial-ordering* (WPO) is a partial ordering $\prec$ for which there is no infinite bad sequence.

If $\prec_1$, $\prec_2$ are partial orderings, $\prec_1 \times \prec_2$ is the partial ordering on pairs given by

$$(x, y) \prec_1 \times \prec_2 (x', y') \quad \text{iff} \quad x \prec_1 x' \text{ and } y \prec_2 y'.$$

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$\mathcal{I}$ is a model of **ProdWPO** iff whenever $\prec_1$ and $\prec_2$ are partial orderings in $\mathcal{I}$ and there exists an infinite bad sequence in $\prec_1 \times \prec_2$, there exists an infinite bad sequence in either $\prec_1$ or $\prec_2$.

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Suppose $\prec_1 \times \prec_2$ has an infinite bad sequence $(x_1, y_1), (x_2, y_2), \ldots$. We define a coloring

$$c(n,m) = \begin{cases} 1 & \text{if } x_n \prec_1 x_m \text{ (so } y_n \not\prec_2 y_m) \\ 2 & \text{if } y_n \prec_2 y_m \text{ (so } x_n \not\prec_1 x_m) \\ 0 & \text{if } x_n \not\prec_1 x_m \text{ and } y_n \not\prec_2 y_m \end{cases} .$$

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Then **ProdWPO** says that when we have colorings like this—a 3-coloring of pairs such that two of colors transitive—then there is an infinite set H so that $[H]^2$ omits one of the transitive colors.

Theorem (Cholak-Marcone-Solomon and Frittaion-Marcone-Shafer)

Every model of **CAC** *is a model of* **ProdWPO** *and every model of* **ProdWPO** *is a model of* **ADS**.

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Every model of **CAC** *is a model of* **ProdWPO** *and every model of* **ProdWPO** *is a model of* **ADS**.

Theorem (Lerman-Solomon-T.)

There are models of **ADS** *which are not models of* **CAC**.

Question

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Separating **ProdWPO** from **ADS** can probably be done by modifying the LST construction.

In most of these separation arguments, we work with some (possibly approximate) "limit coloring": a point x is colored i in the limit if for "many" y, $c(x, y) = i$.

To separate **ProdWPO** from **CAC**, we would like to make it "**ADS**-like". Since pairs with color 0 always allowed, pairs with color 0 seem like "freebies" we can use anywhere we want. This means trying to pretend that points with limit color 0 actually have limit coloring one of the other, better behaved, colors.

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We naturally end up trying to say "consider all the ways we can replace 0's with either 1 or 2". But this leads us to a finitely branching tree of ways of replacing 0's with 1's or 2's.

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$\mathcal{I}$ is a model of **WKL** if whenever T is an infinite tree of $\{0,1\}$ sequences in $\mathcal{I}$, $\mathcal{I}$ also contains an infinite path through T.

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Question

Does **ADS** + **WKL** *imply* **CAC***?*

Crucial step in the LST construction: we want to create a partial ordering which is difficult to solve. We wish to diagonalize against a particular prospective solution.

- We look for two "blocks" of witnesses (coming from prospective solutions) which must come from disjoint intervals. We restrain one block to be "chain" elements (below cofinitely many elements) and the other to be "antichain" elements (incomparable to cofinitely many elements).
- If we fail to find this, we have to show that the prospective solution is finite.

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- If we fail to find this, we have to show that the prospective solution is finite.

This seems to be incompatible with trying to find solutions to **WKL**. Roughly speaking, we would have to carry this argument out in each branch of a finitely branching tree.

But it might be that we find such a configuration in every branch, but that the first block of witnesses from some branches overlaps the second block of witnesses in other branches.

Theorem (T.)

- **WKL** + **ADS** *does not imply* **ProdWPO**, *and*
- **WKL** + **ProdWPO** *does not imply* **CAC**.

To fix the LST argument, we need to do the following:

- We can only look for one block of witnesses, all of which must be treated identically (i.e. restrained the same way).
- If we fail to find the witnesses, this must also be enough to guarantee that we satisfy the requirement (for instance, by showing that certain sets are finite).

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- We can only look for one block of witnesses, all of which must be treated identically (i.e. restrained the same way).
- If we fail to find the witnesses, this must also be enough to guarantee that we satisfy the requirement (for instance, by showing that certain sets are finite).
- But we could satisfy a requirement in several steps, as long as each step satisfies this dichotomy.
- We can change the restraint of a block based on what step we are at: a block could be restrained one way for a while, and then changed if we find a new block of witnesses to the next step.

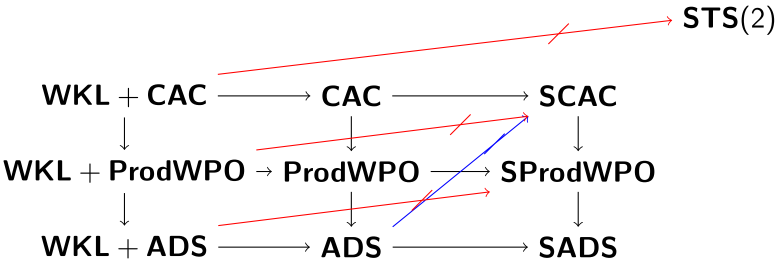
This method is compatible with finding solutions to **WKL** because we can look for the next batch of sequences and witnesses in every branch of a finitely branching tree simultaneously:

- if we fail to find our sequences in a branch, we can continue our construction in that branch,
- if we find sequences in every branch, our block of witnesses is big enough to include all witnesses for every branch,
- we only start looking for the next block of witnesses after every branch has found its witnesses.

Theorem (T. 2016)

- **WKL** + **ADS** *(or even* **WKL**+*"all transitive colorings have solutions") does not imply* **SProdWPO**,
- **WKL** + **ProdWPO** *does not imply* **SCAC**,
- **WKL** + **CAC** *does not imply* **STS**(2).

That is:



Patey identified the notion of *dependent hyperimmunity* and showed that the LST construction corresponds to distinguishing principles which "admit preservation of dependent hyperimmunity".

In particular, **ADS** does admit preservation of dependent hyperimmunity while **CAC** does not.

Question

1. *Does* **ProdWPO** *admit preservation of dependent hyperimmunity?*
2. *Is there an analog of dependent hyperimmunity for the new construction?*

The separation of **ProdWPO** from **CAC** appears to make essential use of **WKL** (or at least **RWKL**).

Question

Is there either:

- *an alternate separation of* **ProdWPO** *from* **CAC** *which does not need* **RWKL**, *or*
- *some way to formally explain why separating* **ProdWPO** *from* **CAC** *involves* **RWKL** *in an essential way.*

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The end.