Weihrauch reducibility and recursion theory

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Computability Theory, Dagstuhl Seminar 2017

Outline

Weihrauch degrees and their structure

Classifying principles from recursion theory

Some open questions and speculations on computable model theory

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Represented spaces and computability

Definition

A *represented space* **X** is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ a surjective partial function.

Definition

$F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $f : \mathbf{X} \rightrightarrows \mathbf{Y}$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \delta_X^{-1}(\mathrm{dom}(F))$.

$$\begin{array}{ccc} \mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

Definition

$f : \mathbf{X} \rightrightarrows \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

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Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, such that $K\langle \mathrm{id}_{\mathbb{N}^\mathbb{N}}, GH\rangle$ is a realizer of f for every realizer G of g.

Definition (Alternative)

For $A, B \subseteq \mathbb{N}^\mathbb{N}$, say that $A \leq_W B$ if $A = \emptyset$ or $\exists n, m$ such that $\forall x \in \mathbb{N}^\mathbb{N}$, if $\exists y \in \mathbb{N}^\mathbb{N}\ \langle x, y\rangle \in A$, then

1. $\Phi_n(x)\downarrow$ and $\exists y \in \mathbb{N}^\mathbb{N}\ \langle \Phi_n(x), y\rangle \in B)$
2. If $\langle \Phi_n(x), y\rangle$ for some $y \in \mathbb{N}^\mathbb{N}$, then $\Phi_m\langle x, y\rangle\downarrow$ and $\langle x, \Phi_m\langle x, y\rangle\rangle \in A$.

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What we know about structure

Theorem (Brattka & Gherardi 2011; P. 2010)

$\mathfrak{W}$ *is a distributive lattice. The cartesian product* $\times$ *is an operation on* $\mathfrak{W}$.

Theorem (Higuchi & P. 2013)

$\mathfrak{W}$ *is not a Brouwer algebra.*

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Embeddings into $\mathfrak{W}$

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For $A \subseteq \mathbb{N}^{\mathbb{N}}$, let $d_A : A \to \{0\}$ and $c_A : \{0\} \rightrightarrows A$. Then $d_. : \mathfrak{M}^{op} \to \mathfrak{W}$ is a lattice embedding and $c_. : \mathfrak{M} \to \mathfrak{W}$ is a meet-semilattice embedding.

Theorem

Let $p, q \in \mathbb{N}^{\mathbb{N}}$ be Turing-incomparable. For $A \subseteq \mathbb{N}$, let $e_A : \mathbb{N} \to \{p, q\}$ map $n \in A$ to p and $n \notin A$ to q. Then $e_.$ is an join-semilattice embedding of the many-one degrees into $\mathfrak{W}$.

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Densely realized problems

Definition (Brattka, Hendtlass & Kreuzer 2015)
Call $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ densely realized, if:

$$\forall p \in \mathbb{N}^\mathbb{N}\ \forall w \in \mathbb{N}^*\ \exists q \in \mathbb{N}^\mathbb{N}\ wq \in f(p)$$

A Weihrauch degree is densely realized, if it has a densely realized representative.

Observation
Problems from recursion theory are typically densely realized.

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The separation

Proposition

Let $f :\subseteq \mathbf{X} \rightrightarrows \mathbb{N}$ and g be densely realized. If $f \leq_W g$, then f is computable.

Definition

Let $\mathrm{ACC}_{\mathbb{N}} : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ be defined via $n \in \mathrm{ACC}_{\mathbb{N}}(p)$, iff $n + 1$ is not the first non-zero entry in p.

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$\mathrm{ACC}_{\mathbb{N}}$ is reducible to every non-computable non-recursion theoretic theorem classified so far.

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Some examples for connections

Theorem (Brattka & P. 2016)

$\mathrm{MLR} \equiv_W \mathrm{C}_{\mathbb{N}} \to \mathrm{WWKL}$.

Theorem (Brattka, Hendtlass & Kreuzer 2015)

1. $\mathrm{COH} \equiv_W \lim \to \mathrm{WKL}'$
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Further reading

V. Brattka, M. Hendtlass and A. Kreuzer.
On the Uniform Computational Content of Computability Theory.
arXiv, 1501.00433, 2015.

Uniform low basis theorem

Definition
Let $\mathrm{L} :\subseteq \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be defined by $q = \mathrm{L}(p)$ iff $H^q = \lim p$.

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Local structure

How does the local structure of the Weihrauch lattice look like?

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Question

What can we say about the interval $[\mathrm{WKL}, \lim]$*?*

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A uniform view on computable model theory I

Observation

Given some finite signature $\mathcal{L}$, there is a represented space $\mathfrak{P}_\mathcal{L}$ of countable presentations of $\mathcal{L}$-structures.

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Define $\mathrm{Iso}_\mathcal{L} :\subseteq \mathfrak{P}_\mathcal{L} \times \mathfrak{P}_\mathcal{L} \rightrightarrows \mathbb{N}^\mathbb{N}$ via $(A, B) \in \mathrm{dom}(\mathrm{Iso}_\mathcal{L})$ iff $A \cong B$, and $p \in \mathrm{Iso}_\mathcal{L}(A, B)$ if p is an $\mathcal{L}$-isomorphism from A to B. Let $\mathrm{Iso}^A_\mathcal{L}$ be the restriction to presentations of A.

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A uniform view on computable model theory II

Claim: The Weihrauch degree of $\mathrm{Iso}_{\mathcal{L}}^{A}$ tells us (almost?) everything we want to know about the degree of categoricity of A.

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$\mathrm{Iso}_{\mathcal{L}}^{A} \equiv_W \lim$ *is the uniform version of saying that A has strong degree of categoricity $\emptyset'$. $\widehat{\mathrm{Iso}_{\mathcal{L}}^{A}} \equiv_W \lim$ should capture having degree of categoricity $\emptyset'$.*

Conjecture

$\widehat{\mathrm{Iso}_{\mathcal{L}}^{A}} \equiv_W (\mathrm{Iso}_{\mathcal{L}}^{A})^n$ *iff the spectral dimension of A is at most n.*

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