From equivalence structures to topological groups

Alexander Melnikov

Dagstuhl, Feb 2017.

Suppose $\mathcal{K}$ is a class of (computable) algebraic structures.

What would be considered a "good" classification of structures in $\mathcal{K}$?

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Very few classes admit a Friedberg enumeration.

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- Structure and Anti-structure theorems (Goncharov and Knigh)
- Effective classification of computable structures (MillerR., Lange, and Steiner)
- Effectively closed sets and enumerations (Brodhead and Cenzer)
- Theory of numberings (A book by Ershov)
- PhD Dissertation (Ospichev, in Russian)

Question (Goncharov and Knight 2002)

Is there a Friedberg enumeration of the class of computable equivalence structures?

Goncharov and Knight conjectured that the answer is **NO** because **the invariants are too complicated**.

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Guessing isomorphism $E \cong F$ between eq. structures is a

Π^0_4-complete problem.

Compare this to c.e. sets where $W_e = W_j$ is Π^0_2.

Earlier attempts by Goncharov and Knight, and by Miller R., Lange, and Steiner.

Theorem (Downey, M., Ng)

There **exists** a Friedberg enumeration of computable eq. structures.

This is a non-uniform $0'''$.

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A structure is **computably categorical** if it has a unique computable copy, up to computable isomorphism.

Problem (Maltsev, in the 1960-s)

Describe computably categorical abelian groups.

We have nice satisfactory classifications for:

- p-groups (Smith, indep. Goncharov)
- torsion-free (Nurtazin)
- infinite rank (Goncharov)

Missing cases:

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Case of study: Torsion abelian groups.

What would be considered a "good" classification of c.c. torsion abelian groups?

Theorem (M. and Ng)

There exists a $\mathcal{L}^c_{\omega_1\omega}$ Π^c_4-sentence Ψ such that

$$A \models \Psi \iff A \text{ is a c.c. torsion abelian group.}$$

Furthermore, Π^c_4 is the optimal complexity. (The index set is Π^0_4-complete.)

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- Π^c_4-harness of the index set is the easy(er) part.
- Ψ relies on several subtle **algebraic reductions.**
- Ψ says that a certain diagonalization attempt on **equivalence structures** must fail.
- The analysis of **computable equivalence structures** is in the (scary) combinatorial core of the proof.

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Would my academic semi-grate grate grandfather be happy?

From computable groups to Polish groups

Definition

A computable Polish group is a computable Polish (metric) space equipped with computable group operations.

We consider Polish groups up to topological isomorphism.

Suppose K is a natural class of Polish groups (e.g., connected compact groups).

Can we classify members of K?

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Theorem (M. and Khoussainov)

1. The index sets of **profinite** and of **connected compact** Polish groups are Arithmetical.
2. The topological isomorphism problems for **profinite abelian groups** and for **connected compact abelian** groups are Σ^1_1-complete.

We can list all partial computable Polish groups: $G_0, G_1, G_2, \ldots$

- $\{i : G_i$ is a connected topological group$\}$ is Arithmetical.
- $\{(i,j) : G_i \cong G_j$ and G_i, G_j are connected$\}$ is Σ^1_1-complete.

The result is uniform. It follows connected and profinite (abelian) groups are **unclassifiable**.

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- Computable Polish space theory.
- **Computable (countable) abelian group theory** (e.g., the old result of Dobrica on bases, a result of Downey and Montalban, etc.).
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(If there is time.)

Definition

Let $\mathbb{T}$ be the unit circle group. The **dual group** of a topological group G is

$$\widehat{G} = \{\chi \mid \chi \text{ is a continuous group homomorphism from } G \text{ to } \mathbb{T}\}.$$

Theorem (Pontryagin)

Let G be either discrete or compact abelian group. Then:

- $\widehat{\widehat{G}} \cong G$, and
- G is compact iff G is discrete.
- G is **torsion** iff $\widehat{G}$ is **profinite**.

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A profinite group is *recursive* if it is the limit of a computable surjective inverse system of finite groups.

($\widehat{G}$ stands for the Pontryagin dual of G.)

Theorem (Khoussainov and M.)

Let G be a countable torsion abelian group. Then

- G is computable iff $\widehat{G}$ is a recursive profinite group;
- G is computably categorical iff $\widehat{G}$ is computably categorical (as a recursive profinite group).

Corollary (follows from M. and Ng)

The index set of c.c. recursive profinite groups is Π^0_4-complete.

eq. structures $\rightarrow$ (discrete) abelian groups $\rightarrow$ Polish groups.

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