A peek at the higher levels of the Weihrauch hierarchy

Alberto Marcone
(work in progress with Andrea Cettolo)

Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Università di Udine
Italy
alberto.marcone@uniud.it
http://www.dimi.uniud.it/marcone

Computability Theory
February 19–24, 2017
Schloss Dagstuhl

Outline

1. Weihrauch reducibility

2. The higher levels of the Weihrauch hierarchy

TTE computability

TTE Turing machines have one input tape, one working tape and one output tape and each tape has a head.

All ordinary instructions for Turing machines are allowed for the working tape, while the head of the input tape can only read and move forward, and the head of the output tape can only write and move forward.

Hence they cannot correct the output: once a digit is written, it cannot be canceled or changed. This means that each partial output is reliable.

TTE Turing machines can be viewed as ordinary oracle Turing machines: the oracle supplies the information about the input and the n-th bit of the output is computed when we give n as input to the oracle Turing machine.

The partial functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$ computed by TTE machines are the Lachlan functionals: we call them computable partial functions from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$.

Represented spaces

A representation σ_X of a set X is a surjective function $\sigma_X : \subseteq \mathbb{N}^{\mathbb{N}} \to X$.

The pair (X, σ_X) is a represented space.

If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

Representations are analogous to the codings used in reverse mathematics to speak about various mathematical objects in subsystems of second order arithmetic.

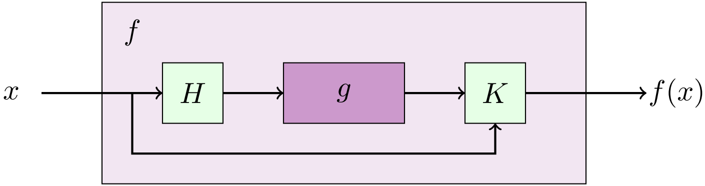
For example computable metric spaces are represented via the Cauchy representation.

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f : \subseteq X \rightrightarrows Y$ we say that f is computable if there exists a computable $F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\sigma_Y(F(p)) \in f(\sigma_X(p))$ whenever $f(\sigma_X(p))$ is defined.

Weihrauch reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions between represented spaces.

f is Weihrauch reducible to g, $f \le_W g$, if there are computable $H :\subseteq X \rightrightarrows Z$ and $K :\subseteq X \times W \rightrightarrows Y$ such that $K(x, gH(x)) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$:



$f \le_W g$ means that the problem of computing f can be computably and uniformly solved by using in each instance a single computation of g: H modifies the input of f to feed it to g, while K, using also the original input, transforms the output of g into the correct output of f.

The Weihrauch hierarchy

$\leq_W$ is reflexive and transitive and induces the equivalence relation $\equiv_W$.

The partial order on the sets of $\equiv_W$-equivalence classes (called Weihrauch degrees) is a distributive bounded lattice with several natural and useful algebraic operations. We call it the Weihrauch hierarchy.

The Weihrauch hierarchy allows a calculus of mathematical problems.

A mathematical problem can be identified with a partial multi-valued function $f :\subseteq X \rightrightarrows Y$: there are sets of potential inputs X and outputs Y, $\mathrm{dom}(f) \subseteq X$ contains the valid instances of the problem, and $f(x)$ is the set of solutions of the problem f for instance x.

If $\forall x \in X(\varphi(x) \to \exists y \in Y \ \psi(x, y))$ is a true statement,
we consider the mathematical problem with domain $\{ x \in X \mid \varphi(x) \}$
such that $f(x) = \{ y \in Y \mid \psi(x, y) \}$.

The Weihrauch hierarchy and reverse mathematics

In most cases the Weihrauch hierarchy refines the classification provided by reverse mathematics: statements which are equivalent over RCA_0 may give rise to functions with different Weihrauch degrees.

Weihrauch reducibility is finer because requires both uniformity and use of a single instance of the harder problem.

There are however exceptions to this phenomena, and in some cases the reverse mathematics approach may detect differences that Weihrauch reducibility misses:

"the computable analyst is allowed to conduct an unbounded search for an object that is guaranteed to exist by (nonconstructive) mathematical knowledge, whereas the reverse mathematician has the burden of an existence proof with limited means" (Gherardi-M 2009).

Jumping in the Weihrauch hierarchy

$\lim : \subseteq (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ maps a sequence in Baire space to its limit.

$\lim$ corresponds to $0'$, and can be iterated.

$\lim$, and its iterates, often show up when dealing with multi-valued functions arising from theorems equivalent to ACA_0.

$\lim$ can be used to define the jump of any multi-valued function.

For example, (the function corresponding to) the Bolzano-Weiestraß Theorem is Weihrauch equivalent to the jump of (the function corresponding to) Weak Kőnig Lemma.

Choice functions

If X is a computable metric space let $\mathcal{A}^-(X)$ be the space of its closed subsets represented by negative information, i.e. by providing a list of basic open balls whose union is the complement of the closed set.

$\mathsf{C}_X :\subseteq \mathcal{A}^-(X) \rightrightarrows X$ is the choice function for X: it picks from a nonempty closed set in X one of its elements.

Already C_2 is noncomputable and, for example, $\mathsf{C}_{2^\mathbb{N}} \equiv_W \mathsf{WKL}$.

$\mathsf{UC}_X :\subseteq \mathcal{A}^-(X) \to X$ is the unique choice function for X: it picks from a singleton (represented as a closed set) in X its unique element.

UC_2 is computable and, for example, $\mathsf{UC}_\mathbb{N} \equiv_W \mathsf{UC}_\mathbb{R} \equiv_W \mathsf{C}_\mathbb{N}$.

It will be important for us that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_\mathbf{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$

State of the art

In the last decade many (functions arising from) theorems provable in ACA_0 have been classified in the Weihrauch hierarchy.

This study has e.g. helped clarify the relationships between different forms of Ramsey Theorem.

Much less is known about (functions arising from) theorems which lie around ATR_0 and $\mathbf{\Pi}^1_1$-CA_0.

In September 2015 I proposed to start this study during the open problems session of the Dagstuhl seminar “Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis”.

Three functions arising from theorems equivalent to ATR_0

Tr is the space of subtrees of $\mathbb{N}^{<\mathbb{N}}$;
WO is the space of well-orders on $\mathbb{N}$.

$\mathsf{PTr} :\subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr}$ is the multi-valued function that maps a tree with uncountably many paths to the set of its perfect subtrees.

This is not the only possible function arising from the Perfect Tree Theorem (Kihara and Pauly started looking at other functions).

$\mathsf{CWO} :\subseteq \mathrm{WO} \times \mathrm{WO} \to \mathbb{N}^{\mathbb{N}}$ is the function that maps a pair of well-orders to the order preserving map from one of them onto an initial segment of the other.

$\mathsf{WCWO} :\subseteq \mathrm{WO} \times \mathrm{WO} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is the multi-valued function that maps a pair of well-orders to the set of order preserving maps from one of them to the other.

Some functions arising from statements around ATR_0

$\boldsymbol{\Sigma}^1_1$-Sep $: \subseteq (\mathrm{Tr} \times \mathrm{Tr})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ has domain $\{ (S_n, T_n)_{n \in \mathbb{N}} \mid \forall n([S_n] = \emptyset \lor [T_n] = \emptyset) \}$ and maps $(S_n, T_n)_{n \in \mathbb{N}}$ to

$$\{ f \in 2^{\mathbb{N}} \mid \forall n([S_n] \neq \emptyset \to f(n) = 0) \land ([T_n] \neq \emptyset \to f(n) = 1) \}.$$

$\boldsymbol{\Delta}^1_1$-CA is the restriction of $\boldsymbol{\Sigma}^1_1$-Sep to $\{ (S_n, T_n)_{n \in \mathbb{N}} \mid \forall n([S_n] = \emptyset \leftrightarrow [T_n] \neq \emptyset) \}$.

$\boldsymbol{\Delta}^1_1$-CA$^-$ is the restriction of $\boldsymbol{\Delta}^1_1$-CA to $\{ (S_n, T_n)_{n \in \mathbb{N}} \mid \forall n \, |[S_n]| + |[T_n]| = 1 \}$.

$\boldsymbol{\Sigma}^1_1$-CA$^-$ $: \subseteq \mathrm{Tr}^{\mathbb{N}} \to 2^{\mathbb{N}}$ has domain $\{ (T_n)_{n \in \mathbb{N}} \mid \forall n \, |[T_n]| \leq 1 \}$ and maps $(T_n)_{n \in \mathbb{N}}$ to the characteristic function of $\{ n \in \mathbb{N} \mid |[T_n]| = 1 \}$.

Our peek, so far

Theorem (Dagstuhl 2015)

$\mathsf{PTr} \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Theorem

$\mathsf{CWO} \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathbf{\Sigma}^1_1\text{-}\mathsf{Sep} \equiv_W \mathbf{\Delta}^1_1\text{-}\mathsf{CA} \equiv_W \mathbf{\Delta}^1_1\text{-}\mathsf{CA}^- \equiv_W \mathbf{\Sigma}^1_1\text{-}\mathsf{CA}^-$.

It is obvious that $\mathsf{WCWO} \leq_W \mathsf{CWO}$.

Proposition

$\lim^{(k)} <_W \mathsf{WCWO}$ *for every* $k \in \mathbb{N}$.

Question

$\mathsf{WCWO} \equiv_W \mathsf{CWO}$?

Some remarks about the proofs

- CWO $\leq_{\mathrm{W}}$ UC$_{\mathbb{N}^{\mathbb{N}}}$ is straightforward;
- to prove $\mathbf{\Sigma}^1_1$-Sep $\leq_{\mathrm{W}}$ CWO we follow the ideas of the reverse mathematics proofs in Simpson's book, but we need extra care to avoid using CWO more than once;
- the same proof shows $\mathbf{\Sigma}^1_1$-Sep $\leq_{\mathrm{W}} \widehat{\mathsf{WCWO}}$: thus $\widehat{\mathsf{WCWO}} \equiv_{\mathrm{W}}$ UC$_{\mathbb{N}^{\mathbb{N}}}$;
- the most complex proof is the one showing that $\mathbf{\Sigma}^1_1$-CA$^- \leq_{\mathrm{W}} \mathbf{\Delta}^1_1$-CA;
- the other equivalences with UC$_{\mathbb{N}^{\mathbb{N}}}$ are fairly easy;
- $\lim \leq_{\mathrm{W}}$ WCWO follows the idea of Friedman-Hirst's proof that WCWO (as a statement) implies ACA$_0$: we compare a well-order of order type ω with a fixed well-order of order type $\omega + 1$;
- to prove that $\lim^{(k)} \leq_{\mathrm{W}}$ WCWO we compare a well-order of order type $\sum_{i=0}^{k-1} \omega^{k-i}$ with a fixed well-order of order type $\sum_{i=0}^{k} \omega^{k-i}$.

What we learnt so far

- Some theorems equivalent to ATR_0 give rise to functions Weihrauch equivalent to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, others to functions Weihrauch equivalent to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$: as expected the Weihrauch hierarchy can refine the reverse mathematics results;
- some theorems (e.g. comparability and weak comparability of well-orders) give rise to a single natural function, others (e.g. the Perfect Tree Theorem) to several functions that are likely to have different Weihrauch degree;
- some functions arising from statements properly weaker than ATR_0 are Weihrauch equivalent to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$: here it is probably important the fact that the domain of some functions is not Borel.