The Scott Isomorphism Theorem

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Formulas of $L_{\omega_1\omega}$

$L_{\omega_1\omega}$ formulas are infinitary formulas in which the disjunctions and conjunctions are over countable sets, and the strings of quantifiers are finite.

We consider only formulas with finitely many free variables. Bringing the negations inside, we get a kind of *normal form*. Formulas in this normal form are classified as Σ_α or Π_α, for $\alpha < \omega_1$.

Classification

1. $\varphi(\bar{x})$ is Σ_0 and Π_0 if it is finitary quantifier-free,
2. for $0 < \alpha < \omega_1$,
   (a) $\varphi(\bar{x})$ is Σ_α if it is a countable disjunction of formulas $(\exists\bar{u})\psi(\bar{x},\bar{u})$, where $\psi(\bar{x},\bar{u})$ is Π_β for some $\beta < \alpha$,
   (b) $\varphi(\bar{x})$ is Π_α if it is a countable conjunction of formulas $(\forall\bar{u})\psi(\bar{x},\bar{u})$, where $\psi(\bar{x},\bar{u})$ is Σ_β for some $\beta < \alpha$.

Computable infinitary formulas

For a computable language L, the *computable infinitary L-formulas* are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are over c.e. sets.

These formulas are classified as computable Σ_α or computable Π_α for $\alpha < \omega_1^{CK}$.

Sample formulas

(1) Suppose G is a group generated by $\bar{a}$. To say that $\langle \bar{x} \rangle \cong \langle \bar{a} \rangle$, we take the Π_1 formula

$$\bigwedge_{G \models w(\bar{a})=e} w(\bar{x}) = e \;\&\; \bigwedge_{G \models w(\bar{a}) \neq e} w(\bar{x}) \neq e \; .$$

The formula is computable Π_1 if the group G is computable.

(2) To say that $\bar{x}$ generates G, we take the computable Π_2 formula

$$(\forall y) \bigvee_{w} w(\bar{x}) = y \; .$$

Scott Isomorphism Theorem

Scott (1964). Let $\mathcal{A}$ be a countable structure for a countable language L. Then there is a sentence of $L_{\omega_1\omega}$, a *Scott sentence*, whose countable models are just the isomorphic copies of $\mathcal{A}$.

To obtain a Scott sentence, Scott first found formulas $\varphi_{\bar{a}}(\bar{x})$ that define the orbits of the tuples $\bar{a}$ in $\mathcal{A}$. He then took the conjunction of the following:

$$\rho_{\emptyset} = (\forall x) \bigvee_b \varphi_b(x) \ \& \ \bigwedge_b (\exists x) \varphi_b(x)$$

$$\rho_{\bar{a}} = (\forall \bar{u})[\varphi_{\bar{a}}(\bar{u}) \rightarrow ((\forall x) \bigvee_b \varphi_{\bar{a},b}(\bar{u}, x) \ \& \ \bigwedge_b (\exists x) \varphi_{\bar{a},b}(\bar{u}, x))]$$

The complexity of an optimal Scott sentence measures the internal complexity of $\mathcal{A}$.

Scott rank

There are different definitions of Scott rank.

Montalbán. The *Scott rank* of $\mathcal{A}$ is the least α s.t. $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

Theorem (Montalbán). $\mathcal{A}$ has Scott rank α iff the orbits of all tuples in $\mathcal{A}$ are defined by Σ_α formulas (with no parameters).

I will return to this result.

Possible Scott ranks for computable structures

Nadel (1974). For a computable structure $\mathcal{A}$, $SR(\mathcal{A}) \leq \omega_1^{CK} + 1$.

There are well-known examples of computable structures with various computable ranks. The "Harrison ordering", of type $\omega_1^{CK}(1 + \eta)$, has rank $\omega_1^{CK} + 1$. There are computable structures of rank ω_1^{CK}.

Scott rank ω_1^{CK}

Until recently, the known examples of computable structures of rank ω_1^{CK} were all obtained from a certain kind of tree. The computable infinitary theory was $\aleph_0$-categorical.

Question (Millar-Sacks, 2008, Calvert-Goncharov-K-Millar, 2009). Is there a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical?

Harrison-Trainor-Igusa-K. Yes.

Approximations

Definition. A computable structure $\mathcal{A}$ of high Scott rank is *computably approximable* if every computable infinitary sentence true in $\mathcal{A}$ is also true in some computable structure of low Scott rank.

Question (Goncharov-K, 2002). Is every computable structure of non-computable rank computably approximable?

Harrison-Trainor. No.

Harrison-Trainor

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Definition. For an $L_{\omega_1\omega}$ sentence φ, the *Scott spectrum* is the set of Scott ranks of countable models of φ.

Harrison-Trainor. Assuming PD, S is the Scott spectrum of a sentence of $L_{\omega_1\omega}$ iff there is a Σ^1_1 class $\mathcal{C}$ of linear orderings for which one of the following holds:

1. S is the set of well-founded parts of orderings in $\mathcal{C}$
2. S is the set of orderings obtained from those in $\mathcal{C}$ by collapsing the non-well-founded part to a single element,
3. S is union of the sets in (1) and (2).

Return to Montalbán

Montalbán. $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence iff the orbits of all tuples are defined by Σ_α formulas.
$\Leftarrow$: If the formulas $\varphi_{\bar{a}}$ are Σ_α, then the Scott sentence given by Scott is $\Pi_{\alpha+1}$.

$\Rightarrow$: We use a special kind of "consistency property".

Consistency properties

Makkai. A *consistency property* is a non-empty countable set $\mathcal{C}$ of finite sets S of sentences obtained by substituting constants from ω for the free variables in an $L_{\omega_1\omega}$ formula, s.t.

1. for $S \in \mathcal{C}$, if $\varphi \in S$, then for each sub-formula $\psi(\bar{x})$ of φ, and each $\bar{c}$, there exists $S' \supseteq S$ with $\psi(\bar{c})$ or $neg(\psi(\bar{c}))$ in S',
2. for $S \in \mathcal{C}$, if $\varphi \in S$, where $\varphi = \bigwedge_i (\forall \bar{u}_i)\varphi_i(\bar{u}_i)$, then for each i and $\bar{c}$, there exists $S' \supseteq S$ with $\varphi_i(\bar{c}) \in S'$,
3. for $S \in \mathcal{C}$, if $\varphi \in S$, where $\varphi = \bigvee_i (\exists \bar{u}_i)\varphi_i(\bar{u}_i)$, then for some i and $\bar{c}$, there exists $S' \supseteq S$ with $\varphi_i(\bar{c}) \in S'$,
4. for $S \in \mathcal{C}$, for each finitary quantifier-free L-formula $\varphi(\bar{x})$ and $\bar{c}$, there exists $S' \supseteq S$ with $\pm\varphi(\bar{c}) \in S'$.

Added condition

Let φ be a $\Pi_{\alpha+1}$ Scott sentence for $\mathcal{A}$, $\varphi = \bigwedge_i (\forall \bar{u}_i)\varphi_i(\bar{u}_i)$, where $\varphi_i(\bar{u}_i) = \bigvee_j (\exists \bar{v}_{i,j})\psi_{i,j}(\bar{u}_i, \bar{v}_{i,j})$. A consistency property satisfying the following added condition yields a model of φ.

5. for $S \in \mathcal{C}$, for each i and appropriate $\bar{c}$, there exist j, $\bar{d}$, and $S' \supseteq S$ with $\psi_{i,j}(\bar{c}, \bar{d}) \in S'$.

Montalbán's $\mathcal{C}$ consists of finite sets S of sentences, each Σ_β or Π_β for some $\beta < \alpha$, s.t. some interpretation of the constants in $\mathcal{A}$ makes the sentences true.

Impossible condition

Suppose for a tuple $\bar{a}$ in $\mathcal{A}$, we try to add the following condition.

6. For each appropriate $\bar{c}$, there is a Π_α formula $\psi(\bar{x})$ true of $\bar{a}$ s.t. for some $S' \supseteq S$, $neg(\psi(\bar{c})) \in S'$.

Now, it is no longer true that for each $S \in \mathcal{C}$, there is an interpretation of the constants that makes the sentences true in $\mathcal{A}$. For some $S \in \mathcal{C}$, with conjunction $\chi(\bar{c}, \bar{d})$, we have

$$\mathcal{A} \models (\forall \bar{x})[(\exists \bar{u})\chi(\bar{x}, \bar{u}) \rightarrow \psi(\bar{x})]$$

for all Π_α formulas ψ true of $\bar{a}$.

Defining orbits

For each $\bar{a}$, let $\varphi_{\bar{a}}(\bar{x})$ be a Σ_α formula generating the complete Π_α-type of $\bar{a}$.

We can show that the family $\mathcal{F}$ of finite functions taking $\bar{a}$ to $\bar{b}$ satisfying $\varphi_{\bar{a}}$ has the back-and-forth property.

Hence, for each $\bar{a}$, $\varphi_{\bar{a}}(\bar{x})$ defines the orbit. This proves Montalbán's Theorem.

Effective version of Montalbán

Alvir-K-McCoy. If $\mathcal{A}$ has a computable $\Pi_{\alpha+1}$ Scott sentence, then the orbits of all tuples are defined by computable Σ_α formulas.

The sentences in $S \in \mathcal{C}$ are substitution instances of sub-formulas of the Scott sentence, or finitary quantifier-free.

Note: There is a computable structure $\mathcal{A}$ s.t. the orbits of all tuples are defined by finitary quantifier-free formulas, but there is no computable Π_2 sentence.

A. Miller

A. Miller (1983). If $\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence and a $\Pi_{\alpha+1}$ Scott sentence, then it has a d-Σ_α Scott sentence.

This is based on a result of D. Miller showing that if A and B are disjoint $\Pi_{\alpha+1}$ subsets of $Mod(L)$, closed under isomorphism, then there is a separator that is d-Σ_α. Moreover, the separator can be taken to be closed under isomorphism.

We get the effective version of A. Miller's result.

Alvir-K-McCoy. If $\mathcal{A}$ has a computable $\Sigma_{\alpha+1}$ Scott sentence and a computable $\Pi_{\alpha+1}$ Scott sentence, then it has a computable d-Σ_α Scott sentence.

Finitely generated groups

K-Saraph.

1. Every finitely generated group has a Σ_3 Scott sentence.
2. Every computable finitely generated group has a computable Σ_3 Scott sentence.

Proof.

Let G be generated by the n-tuple $\bar{a}$. We write

$$(\exists \bar{x})[\langle \bar{x} \rangle \cong \langle \bar{a} \rangle \;\&\; (\forall y) \bigvee_{w} w(\bar{x}) = y]$$

□

Can we do better?

I had conjectured the following.

Conjecture 1. Every finitely generated group has a d-Σ_2 Scott sentence.

Conjecture 2. For a computable finitely generated group, there is a computable d-Σ_2 Scott sentence.

The conjectures hold for free groups (Carson-Harizanov-Knight-Lange-McCoy-Morozov-Quinn-Safranski-Wallbaum), Abelian groups (Saraph), dihedral groups (Saraph, Raz), and polycyclic, lamplighter and Baumslag-Solitar groups (Ho).

Counterexample

Definition (Harrison-Trainor-Ho). A finitely generated group G is *self-reflexive* if for some generating tuple $\bar{a}$, there exists $\bar{b}$ generating a proper subgroup $H \cong G$ s.t. all existential formulas true of $\bar{b}$ are true of $\bar{a}$.

Harrison-Trainor-Ho.

1. A self-reflective group cannot have a d-Σ_2 Scott sentence.
2. There is a computable self-reflexive group.

Exactly when is there a d-Σ_2 Scott sentence?

Alvir-K-McCoy. For a finitely generated group G, there is a d-Σ_2 Scott sentence iff the orbit of some (every) generating tuple is defined by a Π_1 formula.

Remark: A finitely generated group is self-reflective iff the orbit of some generating tuple is not defined by a Π_1 formula.

Alvir-K-McCoy. For a computable finitely generated group G, there is a computable d-Σ_2 Scott sentence iff the orbit of some (every) generating tuple is defined by a computable Π_1 formula.

Proof of first result

Let G be finitely generated. We show that G has a d-Σ_2 Scott sentence iff the orbit of some (every) generating tuple is defined by a Π_1 formula.

(1) $\Rightarrow$ **(2)**: If there is a d-Σ_2 Scott sentence, then there is a Π_3 Scott sentence. By Montalbán, the orbit of every tuple is defined by a Σ_2 formula $\varphi(\bar{x}) = \bigvee_i (\exists \bar{u}_i)\varphi_i(\bar{x}, \bar{u}_i)$. If $\bar{a}$ is a generating tuple, and $G \models \varphi_i(\bar{a}, \bar{b})$, then we have $G \models \bar{b} = \bar{w}(\bar{a})$. The orbit of $\bar{a}$ is defined by the Π_1 formula $\varphi_i(\bar{x}, \bar{w}(\bar{x}))$.

(2) $\Rightarrow$ **(1)**: If the orbit of some generating tuple $\bar{a}$ is defined by a Π_1 formula, then for each $\bar{b}$, the orbits of $\bar{b}$ is defined by a Σ_2 formula. By Montalbán, there is a Π_3 Scott sentence. By A. Miller, there is a d-Σ_2 Scott sentence.

Alternative to A. Miller

Ho. For a finitely generated group G, if there is a Σ_2 formula $\varphi(\bar{x})$, satisfied in G, and s.t. every tuple satisfying $\varphi(\bar{x})$ generates G, then G has a d-Σ_2 Scott sentence.

Ho proved the effective version.

The typical finitely generated group

Gromov asked about typical, or random, or average, finitely generated groups. There are different ways to make this precise.

Schupp and Kapovich. Let N_s be the number of presentations of groups, with a fixed n-tuple of generators, and with one relator of length $\leq s$. Let $N_s(P)$ be the number of these presentations s.t. the group has property P. Consider $lim_{s \to \infty} \frac{N_s(P)}{N_s}$.

If the limit exists and is 1, then we say that the typical group on n generators and with one relator has property P.

Theory of free groups

By results of Sela, all non-Abelian free groups have the same elementary first order theory T_F.

Conjecture. For each $n \geq 2$, and each finitary sentence φ in the language of groups, then the typical group on n generators and with 1 relator satisfies φ if $\varphi \in T_F$.

The conjecture says that if P is the property of satisfying φ, then

$$\lim_{s \to \infty} \frac{N_s(P)}{N_s}$$

exists, with value 0 or 1, and the value is 1 just in case $\varphi \in T_F$.