Finding bases of free abelian groups is hard

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The motivating question:

Can you build a basis of a free abelian group by recursion?

Building bases by recursion

We know how to build bases of vector spaces by recursion: take the next vector not in the span of what you have so far.
Uncountable dimension does not present any difficulty: keep doing this transfinitely.

What about bases of $\mathbb{Z}^\kappa$?

P-independence

There are maximal linearly independent subsets of $\mathbb{Z}^\kappa$ which are not bases (even true for $\kappa = 1, 2, \ldots, \omega$).

A necessary condition to be part of a bases is P-independence: generating a pure subgroup.

Fact (Pontryagin)

Suppose that $G = \mathbb{Z}^\kappa$ is free, and that $B \subset G$ is P-independent and finite. Then for all $g \in G$ there is a finite, P-independent $B' \supseteq B$ such that $g \in \langle B' \rangle$.

Corollary (Downey,Melnikov)

Every countable free abelian group G has a $\Delta^0_2(G)$ basis.

In other words: free abelian groups are Δ^0_2-categorical.

Getting stuck at limit stages

However:

Fact

There is a P-independent subset of $\mathbb{Z}^\omega$ *which is not extendible to a basis.*

Is there perhaps a stronger property which will guarantee that things don't go wrong at limit stages?

Theorem

If κ *is regular and uncountable, it is impossible to definably / effectively pass from the group table of* $\mathbb{Z}^\kappa$ *to a basis.*

Background

How is this formalised?

κ-recursion / computability theory.

- We assume that $V = L$.

The universe is $L_\kappa = H_\kappa$. The κ-c.e. subsets of L_κ are those which are $\Sigma_1(L_\kappa)$-definable. Equvalently, those which are enumerated by Turing machines with tape of length κ running for κ many stages. (Ordinal parameters permitted.)

This gives rise to: κ-computable (partial and total functions, sets); definition by recursion; the fixed point theorem; etc.

κ-computable structure theory

Project (Greenberg, Knight): understand κ-effective properties of structures of size κ.

Proposition (Turetsky,Westrick)

ℓ_∞ *is* $\aleph_1$*-computably categorical.*

Other work:

- linear orderings (Greenberg,Kach,Lempp,Turetsky)
- fields (Fokina,Friedman,Knight,R.Miller)
- vector spaces (Greenberg,Knight)
- computable categoricity (Greenberg,Knight,Melnikov,Turetsky)
- trees (Johnston)

Set-theoretic work

Set-theorists looked at two main questions:

- For which cardinals κ, are there κ-free groups which are not κ^+-free?
- Is being free definable in infinitary logic (e.g. $\mathcal{L}_{\kappa,\kappa}$)?

Work by Paul Hill, Eklof, Mekler, Shelah. Answers depend on the set-theoretic universe.

A key theorem is Shelah's singular compactness theorem: if κ is singular, then a group of size κ is free if and only if it is κ-free.

Theorem (Magidor,Shelah)

It is consistent that such compactness holds for $\kappa = \aleph_{\omega^2+1}$. This is the least possible.

Some results

No upper bound

Theorem (with Shelah for the weakly compact case)

If κ is regular and uncountable, then for any $\Delta^1_1(L_\kappa)$ set X there is a κ-computable copy of $\mathbb{Z}^\kappa$ with no X-computable basis.

Proposition (Johnston)

Every κ-computable copy of $\mathbb{Z}^\kappa$ has a $\Delta^1_1(L_\kappa)$ basis.

No lower bound

Is the reason we cannot get simple bases, that we can code complicated information into bases?

No.

Theorem

Depending on the kind of regular cardinal κ, we can either code $\varnothing'$ or $\varnothing''$ into all bases of some κ-computable copy of $\mathbb{Z}^\kappa$. This is optimal.

In particular: if $X \nleqslant_\kappa \varnothing''$ then every κ-computable copy of $\mathbb{Z}^\kappa$ has a basis which does not compute X.

The singular case

Not much is known about singular of uncountable cofinality.

Theorem

Every $\aleph_\omega$-computable copy of $\mathbb{Z}^{\aleph_\omega}$ has a $\emptyset'$-computable basis.

Identifying free groups

Theorem
If κ is regular and not weakly compact, then the collection of free abelian groups (on κ) is $\boldsymbol{\Sigma}^1_1$-complete.

If κ is a successor cardinal, or (for example) the least inaccessible cardinal, then we get Σ^1_1-completeness (no parameter). In these cases, the no-upper-bound on complexity of bases follows.

Theorem
If κ is weakly compact, then the collection of free abelian groups is $\boldsymbol{\Pi}^0_1$-complete; the index set of κ-computable free abelian groups is Π^0_2-complete.

Some tools

Bases and clubs

From now, fix a regular uncountable cardinal κ.

Bases of $\mathbb{Z}^\kappa$ are equicomputable with closed unbounded subsets.

Theorem (Fokina,Friedman,Knight,R.Miller for $\kappa = \aleph_1$)

If κ is not weakly compact then the club filter on κ is $\mathbf{\Sigma}^1_1(L_\kappa)$-complete.

Filtrations

A κ-filtration of a group G is a sequence $\langle G_\alpha \rangle_{\alpha \leqslant \kappa}$ such that:

- $G = G_\kappa$;
- it is increasing: if $\alpha < \beta \leqslant \kappa$ then $G_\alpha \subseteq G_\beta$;
- it is continuous: if $\beta \leqslant \kappa$ is a limit ordinal, then $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$;
- for all $\alpha < \kappa$, $|G_\alpha| < \kappa$.

For a filtration $\bar{G} = \langle G_\alpha \rangle$ we let

$$\operatorname{Div}(\bar{G}) = \{\alpha < \kappa \,:\, \forall \beta \in (\alpha, \kappa)\ (G_\alpha | G_\beta)\}\,.$$

Fact

Let $\langle G_\alpha \rangle$ be a filtration of G; suppose that for all $\alpha < \gamma$, G_α is free. Then G is free if and only if $\operatorname{Div}(\bar{G})$ contains a club.

Indeed, bases of G are equicomputable with club subsets of $\operatorname{Div}(\bar{G})$.

Obtaining the no-lower-bounds

Fix a κ-computable copy G of $\mathbb{Z}^\kappa$, and a κ-computable κ-filtration $\bar{G}$.

The set $\mathrm{Div}(G)$ is Π^0_2.

Theorem

If $X \nleqslant_\kappa \mathrm{Div}(G)$ then there is a basis of G which does not κ-compute X.

Proof.

Baumgartner-Harrington-E.Kleinberg introduced "shooting a club" through a stationary set. We use an effective version of this notion of forcing.

The fact that $\mathrm{Div}(G)$ contains a club allows us to show that this notion of forcing is κ-strategically closed. This allows us to build generic club subsets of $\mathrm{Div}(G)$. □

Σ^1_1-completeness

Suppose that κ is a successor cardinal.

Given a Π^1_1 statement ψ, we want to build a group G on κ which is free if and only if ψ fails in L_κ. By the completeness of the club filter, we obtain a set $U = U_\psi \subseteq \kappa$ which is stationary if and only if ψ holds in L_κ. We want to build G so that $U = \mathrm{Div}(G)^\complement$.

There are two problems:

1. How do we "twist" at ordinals $\alpha \in U$?
2. How do we ensure that each G_α is free?

Deus ex machina

For problem (1), we know how to twist if $\mathrm{cf}(\alpha) = \aleph_0$ (use $\mathbb{Z}^\omega$).

For problem (2), we need to ensure that for all $\alpha < \kappa$, $\mathrm{Div}(\bar{G} \restriction_\alpha)$ contains a club of α: U must be non-reflecting.

Theorem (Jensen, proof of $\square$)

There is a Σ_1 class E which reflects precisely at regular cardinals and contains only ordinals of countable cofinality.

Theorem

If κ is successor, then the nonstationary ideal restricted to subsets of $E \cap \kappa$ is Σ^1_1-complete.

Further, $E \cap \kappa$ is κ-computable.

Inaccessible cardinals

What about limit regular cardinals? Two problems:

1. *E* does reflect at many $\lambda < \kappa$ (the regular ones).
2. $E \cap \kappa$ is not κ-computable, merely κ-c.e.

Solutions:

- We can only do this if κ is $\mathbf{\Pi}^1_1$-describable. We then replace *E* by a non-reflecting subset. This uses the reduction of $\mathbf{\Sigma}^1_1$ to the club filter.
- While we make *G* computable, the filtration $\bar{G}$ is not.

Weakly compact cardinals

Theorem (Jensen)

The following are equivalent (again assuming $V = L$) for an inaccessible cardinal κ:

1. *κ is weakly compact*
2. *Every stationary subset of κ reflects.*

So if κ is weakly compact there is no hope to carry out our plan. And indeed:

Proposition (Mekler and others)

If κ is weakly compact, then an abelian group of size κ is free if and only if every subgroup of size smaller than κ is free.

Corollary

If κ is weakly compact, then the collection of free abelian groups is Π^0_1-complete. *The index set of the κ-computable free abelian groups is Π^0_2-complete.*

Coding

It is not difficult to code $\varnothing'$ into bases of $\mathbb{Z}^\kappa$.

Recall:

- If κ is regular, not weakly compact, then "free abelian" is $\boldsymbol{\Sigma}^1_1(L_\kappa)$.

This implies:

- If κ is a successor and κ^- is regular, not weakly compact, then the collection of free groups in L_κ is $\Sigma^0_1(L_\kappa)$-complete.

In this case given a free group H in L_κ and a Σ^0_1 statement ψ, we can construct a group $G = B(H, \psi)$ in L_κ which is free, and $H|G$ if and only if ψ holds in L_κ. This allows us to code Π^0_2 sets.

Theorem

- *If κ is a successor, and κ^- is regular, not weakly compact, then* $\mathrm{Div}(G)$ *can be Π^0_2-complete, and there is a κ-computable copy of $\mathbb{Z}^\kappa$, all of whose bases compute $\varnothing''$.*
- *If κ is any other regular cardinal, then for any κ-computable copy G of $\mathbb{Z}^\kappa$,* $\mathrm{Div}(G)$ *is $\varnothing'$-computable.*

Questions

Questions

- What about singular cardinals of uncountable cofinality?
- What if $V \neq L$?

Thank you.