Effective Classification and Measure for Countable Structures

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(Partially joint work with Johanna Franklin,
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Question

Among the algebraic field extensions of $\mathbb{Q}$, how many are uniformly computably categorical?

Sub-questions:
- Didn't you mean "computable algebraic field extensions"?
- What's this "uniformly computably categorical"?
- Whadaya mean, "how many"?

Sub-answers:
- No. I mean all algebraic field extensions of $\mathbb{Q}$.
- $A$ is u.c.c. if there is a $\Phi$ s.t., for all $B \sim C = A$ with domain $\omega$, $\Phi(\Delta(B)) \oplus \Delta(C)$ computes an isomorphism from $B$ onto $C$.
- We will put a measure on the space of (isomorphism types of) algebraic field extensions of $\mathbb{Q}$. 
Introduction: a question

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- No. I mean all algebraic field extensions of $\mathbb{Q}$.
- $\mathcal{A}$ is u.c.c. if there is a $\Phi$ s.t., for all $\mathcal{B} \cong \mathcal{C} \cong \mathcal{A}$ with domain $\omega$, $\Phi^{\Delta(\mathcal{B}) \oplus \Delta(\mathcal{C})}$ computes an isomorphism from $\mathcal{B}$ onto $\mathcal{C}$.
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- We will put a measure on the space of (isomorphism types of) algebraic field extensions of $\mathbb{Q}$.
The space of isomorphism types

We work in the language of fields, extended by \( d \)-ary root predicates:

\[
R_d(a_0, \ldots, a_{d-1}) \iff (\exists x) \ x^d + a_{d-1}x^{d-1} + \cdots + a_0 = 0.
\]

Let \( \text{Alg}^\ast \) be the class of all atomic diagrams of algebraic field extensions of \( \mathbb{Q} \), with domain \( \omega \). By Gödel coding, we view these diagrams as elements of \( 2^\omega \), yielding the quotient topological space \( \text{Alg}^\ast / \equiv \).

**Theorem**

There is a computable homeomorphism \( H \) from \( \text{Alg}^\ast / \equiv \) onto \( 2^\omega \).

That is, both \( H \) and \( H^{-1} \) are computed by Turing functionals:

\[
\Phi_{\Delta(F^\ast)} = H([F^\ast]), \quad \text{and} \quad \Psi^J = H^{-1}(J) \in \text{Alg}^\ast \text{ for all indices } J \in 2^\omega, \text{ with}
\]

\[
\Psi\left(\Phi_{\Delta(F^\ast)}\right) \equiv F^\ast \quad \& \quad \Phi(\Psi^J) = J.
\]
Computing this homeomorphism

\[ X^4 - 2 \quad \mathbb{Q}(\sqrt[4]{2}) \quad \mathbb{Q} \]

By a result of Kronecker, all finite algebraic extensions \( \mathbb{Q}(a_1, \ldots, a_n) \) have splitting algorithms, uniformly in \( \vec{a} \). So this tree is computably homeomorphic to \( 2^\omega \).
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List \( \overline{\mathbb{Q}}[X] \)
Computing this homeomorphism

\[
\begin{align*}
X^2 - 4\sqrt{2} & \quad \mathbb{Q}(\sqrt{2}) \quad \mathbb{Q}(\sqrt{2}) \\
X^8 - 2 & \quad \mathbb{Q}(\sqrt{2}) \\
X^2 - 2 & \quad \mathbb{Q}(\sqrt{2}) \\
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By a result of Kronecker, all finite algebraic extensions $\mathbb{Q}(a_1, \ldots, a_n)$ have splitting algorithms, uniformly in $\bar{a}$. So this tree is computably homeomorphic to $2^\omega$. 
Why the root predicates?

One can consider the space $\text{Alg}/\cong$, without the root predicates. This topological space is not homeomorphic to any standard Polish space: it is a *spectral space*, with one element that lies in every non-empty open set, and another element lying in no proper open subset of the space. So it is not useful for classification or measure.
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The space $\text{Alg}'/\cong$ consists of the atomic diagrams of jumps of algebraic fields, modulo $\cong$. Here $[\mathbb{Q}']$ forms a singleton open set, being the only field in which this $\Sigma^c_1$ predicate fails:

$$(\exists p \in \mathbb{Q}[X])(\exists x \in F) [p \text{ irreducible of degree } > 1 \& p(x) = 0].$$

The root predicates constitute just enough information for a classification by $2^\omega$. 

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A measure on $\text{Alg}^*/\simeq$

Since $\text{Alg}^*/\simeq$ is homeomorphic to $2^\omega$, it seems natural to transfer the Lebesgue measure from $2^\omega$ to $\text{Alg}/\simeq$. But this requires care. In our first example, the odds of 2 having a square root were $\frac{3}{4}$. In general the measure depends on the ordering $f_0, f_1, f_2, \ldots$ of $\overline{\mathbb{Q}}[X]$.

Fix a computable $\overline{\mathbb{Q}}$, and enumerate $\overline{\mathbb{Q}}[X] = \{ f_0, f_1, \ldots \}$. Let $F_\lambda = \mathbb{Q}$. Given $F_\sigma \subset \overline{\mathbb{Q}}$, we find the least $i$, with $f_i$ irreducible in $F_\sigma[X]$ of prime degree, for which it is not yet determined whether $f_i$ has a root in $F_\sigma$. Adjoin such a root to $F_\sigma^\lambda 1$, but not to $F_\sigma^\lambda 0$. This gives a homeomorphism from $2^\omega$ onto $\text{Alg}^*/\simeq$, via $h \mapsto \bigcup_n F_{h|n}$.

If we now transfer standard Lebesgue measure to $\text{Alg}^*/\simeq$, we get a measure in which the odds of 2 having a 1297-th root are $\frac{1}{2}$, but the odds of 2 having a 1296-th root are $\frac{1}{256}$. Is this reasonable?
Haar-compatible measure on $\text{Alg}^*/\cong$

A further improvement is to use *Haar-compatible measure* $\mu$ on $\text{Alg}^*/\cong$. Here the probability of $f_\sigma$ having a root is deemed to equal $\frac{1}{\deg(f_\sigma)}$. This idea (and the name) are justified by:

**Proposition**

For every algebraic field $F_0$ which is normal of finite degree $d$ over $\mathbb{Q}$,

$$
\mu([K] \in \text{Alg}/\cong : F_0 \subseteq K) = \frac{1}{d}.
$$

Notice that $\frac{1}{d}$ is precisely the measure of the pointwise stabilizer of $F_0$ within the group $\text{Aut}(\overline{\mathbb{Q}})$, under the Haar measure on this compact group.
Measuring properties of algebraic fields

Use either of these measures. For (the isomorphism type of) an algebraic field, the property of being normal has measure 0. So does the property of having relatively intrinsically computable predicates $R_d$. 
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In $\text{Alg}^*$, the property of being uniformly computably categorical has measure 1: given two roots $x_1, x_2$ of the same irreducible polynomial, one can wait for them to distinguish themselves, since with probability 1 there is an $f$ for which $f(x_1, Y)$ has a root in the field but $f(x_2, Y)$ does not. This allows computation of isomorphisms between copies of the field. The process works uniformly except on a measure-0 set of fields.
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Surprisingly, measure-1-many fields in $\text{Alg}$ remain uniformly computably categorical even when the root predicates are removed from the language. However, the procedures for computing isomorphisms are not uniform. A single procedure can succeed only for measure-$(1 - \epsilon)$-many fields.
Lemma

Let $\alpha \neq \beta \in \overline{\mathbb{Q}}$ be algebraic numbers conjugate over $\mathbb{Q}$. Then, for every finite algebraic field extension $E \supseteq \mathbb{Q}(\alpha, \beta)$, there is a set $D = \{q_0 < q_1 < \cdots \} \subseteq \mathbb{Q}$, decidable uniformly in $E$, such that for every $k$, both of the following hold:

$$\sqrt{\alpha + q_k} \notin E(\sqrt{\alpha + q_l}, \sqrt{\beta + q_l} : l \neq k)(\sqrt{\beta + q_k});$$

$$\sqrt{\beta + q_k} \notin E(\sqrt{\alpha + q_l}, \sqrt{\beta + q_l} : l \neq k)(\sqrt{\alpha + q_k}).$$

The point is that, for all $q$ in this decidable set $D$, a field extending $E$ may contain both $\sqrt{\alpha + q}$ and $\sqrt{\beta + q}$, or just one of them, or neither, no matter which other $\sqrt{\alpha + \tilde{q}}$ and/or $\sqrt{\beta + \tilde{q}}$ it contains (for $\tilde{q} \neq q$ in $D$). The probability of containing $\sqrt{\alpha + q}$ is always $\frac{1}{2}$, independently of all the rest. The proof uses the Hilbert Irreducibility Theorem.
Proving the theorem

Given an $\epsilon > 0$, and a polynomial $f \in \mathbb{Q}[X]$ with roots $\alpha \neq \beta$ in $\overline{\mathbb{Q}}$, fix the set $D$ from the lemma and choose $N$ so large that the odds are $> 1 - \epsilon$ that, in an arbitrary field $\supseteq \mathbb{Q}(\alpha, \beta)$, all of the following hold:

- For at least $0.4N$ of the numbers $q_0, \ldots, q_{N-1}$ in $D$, $\alpha + q_i$ has a square root in the field.
- For at most $0.35N$ of these numbers, $\alpha + q_i$ and $\beta + q_i$ both have square roots in the field.

The procedure for mapping $\alpha, \beta \in F$ to the right images in a copy $\tilde{F}$ waits until at least $0.4N$ elements $\sqrt{\alpha + q_i}$ with $i < N$ have appeared in $F$. Then it maps $\alpha$ to the first $\tilde{\alpha} \in \tilde{F}$ it finds for which corresponding elements $\sqrt{\tilde{\alpha} + q_i}$ all appear in $\tilde{F}$.

For polynomials of larger degree, use a similar procedure considering each possible pair of roots of the polynomial.
Randomness and computable categoricity

**Theorem (Franklin & M.)**

For every Schnorr-random real $h \in 2^\omega$, the corresponding field $F_h$ is uniformly computably categorical, even in the language without the root predicates. However, there exists a Kurtz-random $h$ for which $F_h$ is not u.c.c. (in the language without the root predicates).

This raises a broader idea: we can consider an isomorphism type $[F]$ to be random just if its index in $2^\omega$ is a (ML, Schnorr, etc.)-random set. This appears compatible with Khoussainov’s concept of random structures.

In terms of computable structure theory, adding the root predicates ensures that every $F^*$ has an upper cone as its spectrum. But the homeomorphism is stronger than this: one field may be ML-random and another not, even when both have the same upper cone as their spectra.
Finite-branching trees

For another example, consider the class $\mathcal{T}_1$ of all finite-branching infinite trees, under the predecessor function $P$. As before, we get a topological space $\mathcal{T}_1/\sim$, which is not readily recognizable. The obvious predicates to add are the branching predicates $B_n$:

$$\models_T B_n(x) \iff \exists y \geq n (P(y) = x).$$

The enhanced class $\mathcal{T}_1^*$ again has a nice classification. Let $T_{m,0}, T_{m,1}, \ldots$ list all finite trees of height exactly $m$. Given $T \in \mathcal{T}_1^*$, we can find the unique number $f(0)$ with $T_{1,f(0)} \equiv T^{<2}$, where $T^{<2}$ is just $T$ chopped off after level 1.
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Next consider those trees in $T_{2,0}, T_{2,1}, \ldots$ with $T_{2,i}^{<2} \cong T^{<2}$. Choose $f(1)$ so that $T^{<3}$ is isomorphic to the $f(1)$-th tree on this list. Continue choosing $f(2), f(3), \ldots$ recursively this way.
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Measure for FB trees (joint with Franklin)

One can place probability measures on $\omega^\omega$ by regarding it as the space of all irrational numbers in $(0, 1)$, using continued fractions; or by assigning measure $2^{- (1 + \sigma(0))} \cdots 2^{- (1 + \sigma(|\sigma| - 1))}$ to the set of all paths through $\sigma$. These measures are similar, but both have certain problems.

- In $\mathcal{T}_1^*/\cong$, one must normalize to account for the requirement that the tree be infinite.
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- In $T_{1^*}/\cong$, one must normalize to account for the requirement that the tree be infinite.
- In the class $T_{0^*}/\cong$ of all FB-trees (including finite ones), no normalization is required, but the finite trees form a class of full measure. Boring! (Or maybe not...)
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In all three situations, measure-1-many trees are uniformly c.c. when the branching predicates are included in the language. The interesting question is what happens in the original language.
Computable categoricity for finite-branching trees

In $\mathcal{T}_0$, uniform computable categoricity has full measure, because every finite tree is u.c.c. However, there is no single procedure $\Phi$ which succeeds on measure-0.999-many trees in $\mathcal{T}_0$. 
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one does not know whether to wait for the two left-hand nodes to distinguish themselves: there is positive probability that they will, and also that they will not.
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In $\mathcal{T}_2/\cong$, we conjecture that with the geometric measure, uniform computable categoricity holds on a set of measure 1, but not uniformly. This is somewhat similar to $\text{Alg}/\cong$.

(All work here joint with Franklin.)
Does this always happen?

**Question**

Which classes of countable structures have definable relations which can be Morleyized to make the class (under $\equiv$) homeomorphic to $2^\omega$ or $\omega^\omega$? And, for standard classes, what are those relations?

- A subring $R$ of $\mathbb{Q}$ is determined by the set of primes $p$ with $\frac{1}{p} \in R$. So the class of all these subrings is computably homeomorphic to $2^\omega$, in a language with a unary predicate for invertibility.
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- Equivalence structures have $\Pi^0_4$ isomorphism problem. Several different possible solutions present themselves here.

In general, one may need to settle for classification as $2^\omega / E$, for certain standard equivalence relations $E$ on $2^\omega$ (or on $\omega^\omega$).
TFAb$_n$: Torsion-free abelian groups of rank $n$

TFAb$_n$/\cong has the indiscrete topology.

TFAb$_n^*$ includes divisibility predicates $D_p(x)$ for all primes $p$. Now, for any $n$-tuple of independent elements in $G$, the atomic diagram of $G$ gives us an element of $2^\omega$, describing which elements of $\mathbb{Q}^n$ (with respect to this basis) lie in $G$. 
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Problem: For different choices of basis, we get different \( h \in 2^\omega \). Thus TFAb\(^*\)_\(_n\)/\( \cong \) is computably homeomorphic to \( 2^\omega / E \), for some ER \( E \).

For \( n = 1 \), \( E \) is just \( E_0 \), the relation of finite-symmetric-difference.
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For \( n > 1 \), \( E \leq_0 E_{\text{set}} \), where \( A E_{\text{set}} B \) means that every column \( A[n] \) of \( A \) appears as a column of \( B \) and vice versa. However, \( E_{\text{set}} \) is strictly harder than all of these. By results of Hjorth & Thomas, these \( E \) must get strictly harder (even in Borel reducibility) as \( n \) increases.
**RCF\(_d\): Countable archimedean real closed fields**

(Joint work with Fokina, Friedman, Rossegger, & San Mauro.)

\[ \text{RCF}_d = \{ D \in 2^\omega : D \text{ is an archimedean RCF of trans. degree } d \}. \]
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Two fields in \(\text{RCF}_d\) are isomorphic iff they fill the same cuts in \(\mathbb{R}\).

Baire space \(\omega^\omega\) is computably homeomorphic to the space of all transcendental \(x \in \mathbb{R}\). Let \(g, h\) be \(E\)-equivalent if they generate the same (strict left) Dedekind cuts. For \(d = 1\), we can list out the cuts generated by a given \(g \in \omega^\omega\) (as elements of \(2^\omega\)). Thus \(\text{RCF}_1 / \cong\) is classified by \(\omega^\omega / E\), which computably reduces to \(E_{\text{set}} \) on \(2^\omega\).
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Two fields in \(\text{RCF}_d\) are isomorphic iff they fill the same cuts in \(\mathbb{R}\).

Baire space \(\omega^\omega\) is computably homeomorphic to the space of all transcendental \(x \in \mathbb{R}\). Let \(g, h\) be \(E\)-equivalent if they generate the same (strict left) Dedekind cuts. For \(d = 1\), we can list out the cuts generated by a given \(g \in \omega^\omega\) (as elements of \(2^\omega\)). Thus \(\text{RCF}_1/\cong\) is classified by \(\omega^\omega/E\), which computably reduces to \(E_{\text{set}}\) on \(2^\omega\).

For \(d > 1\), the same procedure works: parse each \(g \in \omega^\omega\) into \(d\) distinct functions \(g^{[i]}\), for \(i < d\), and use \(g^{[i]}\) to define a real number transcendental over the preceding ones.

Questions: Do these get harder as \(d\) increases? And how do they compare with the classes \(\text{TFAb}_n^*/\cong\)?
Properties of $\text{TFAb}_n$ and $\text{RCF}_d$

All these structures (except in $\text{RCF}_\omega$) are relatively computably categorical, but for $n, d > 0$, they all require constants. Categoricity questions are less interesting here.

The $E$-equivalence classes in $2^\omega/E$ (for $\text{TFAb}_n^*$) and $\omega^\omega/E$ (for $\text{RCF}_d$) all have measure 0. So we do get measures on $\text{TFAb}_n^*$ and on $\text{RCF}_d$ from those on Cantor and Baire space.

**Question:** what about randomness? On $\text{TFAb}_1$, the relation $E_0$ respects all reasonable notions of randomness, so it makes sense to define randomness in $\text{TFAb}_1$. It seems likely that this holds in $\text{TFAb}_n$ as well, and in $\text{RCF}_d$ for $d < \omega$. 