Effective inseparability and its applications

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Summary

We survey recent applications of the classical notion of effective inseparability to the study of:

- computably enumerable (c.e.) equivalence relations (ceers) Andrews, Lempp, Miller J. S., Ng, San Mauro and Sorbi [1]
- c.e. pre-ordering relations and pre-ordered structures Andrews and Sorbi [2]
- c.e. Lindenbaum algebras of sentences Pianigiani and Sorbi [13]
- word problems of c.e. algebras Delle Rose, San Mauro and Sorbi [7]

Definition (Smullyan [18])

A disjoint pair (A, B) of sets of natural numbers is effectively inseparable (or, simply, e.i.) if there exists a partial computable function $\psi(u, v)$ (called a productive function for the pair) such that

$$(\forall u, v)[A \subseteq W_u \,\&\, B \subseteq W_v \,\&\, W_u \cap W_v = \emptyset \Rightarrow \\ \psi(u, v)\downarrow \,\&\, \psi(u, v) \notin W_u \cup W_v].$$

Remark

Each half of an e.i. pair of c.e. sets is creative.

Classical results on e.i. pairs of c.e. sets (restriction to c.e. sets)

- Universality (or completeness) of e.i. pairs under 1-reducibility on disjoint pairs, generalizing the Myhill Completeness Theorem for creative sets.
- All e.i. pairs are computably isomorphic (via a computable permutation of ω), generalizing the Myhill Isomorphism Theorem for creative sets.

Applications to formal systems Let T be any consistent c.e. extension of Robinson's system Q. Then

- the pair $(\mathrm{Thm}_T, \mathrm{Ref}_T)$ is e.i. (T is e.i.), where:

$$\mathrm{Thm}_T = \{\alpha \in \mathrm{Sent} : T \vdash \alpha\}$$
$$\mathrm{Ref}_T = \{\alpha \in \mathrm{Sent} : T \vdash \neg\alpha\}$$

- Q is essentially undecidable

Proof.

It is enough to prove that Q is e.i. and then use: if $(A, B), (C, D)$ are disjoint pairs, (A, B) e.i., and $(A, B) \subseteq (C, D)$ then (C, D) is e.i., and thus each of the two halves is undecidable). □

Effective inseparability and universality for computably enumerable equivalence relations

Definition

A nontrivial eqrel E on ω is uniformly effectively inseparable (u.e.i.) if there is a computable function $f(a,b)$ such that if $a \not\mathrel{E} b$ then $\varphi_{f(a,b)}(u,v)$ is productive for the pair of E-classes $([a]_E,[b]_E)$.

Definition (Computable reducibility for binary relations)

Given eqrels R, S (or more generally two binary relations) on ω we say that R is computably reducible to S ($R \leq_c S$) if

$$(\forall x, y)[x \mathrel{R} y \Leftrightarrow f(x) \mathrel{S} f(y)]$$

for some computable function f.

Definition (Universal ceers)

A ceer E is universal if $R \leq_c E$ for every ceer R.

Theorem (Universality of u.e.i. ceers Andrews-et-al. [1])

Every u.e.i. ceer is universal (via a 1*-reduction).*

Proof. By the Recursion Theorem. (Here and in some of the subsequent proofs exploting effective inseparability, the proof can be roughly described as using a decidable infinite list of indices which we simultaneously control by the Recursion Theorem.)

Universality and density for c.e. preordering relations and c.e. preordered structures

Can anything like this be done for c.e. preorders? For instance do we have universality for any c.e. preordering $\leq$ so that its associated equivalence relation ($x \equiv y$ if $x \leq y$ and $y \leq x$) is u.e.i.?

This is not always so, but it is so if we add additional structure to $\leq$ and $\equiv$.

Definition

A c.e. structure A is a nontrivial algebraic-relational structure for which there exists a c.e. presentation, i.e. a structure A_ω of the same type as A but with universe ω and possessing uniformly computable operations, uniformly c.e. relations, and a ceer $=_A$ which is a congruence on A_ω such that A is isomorphic with A_ω divided by $=_A$ (i.e. $A \simeq A_{\omega/=_A}$).

Remark When talking about a c.e. structure A in the following we intend in fact a c.e. presentation A_ω of the structure. See Selivanov [15] for a great introduction to c.e. structures.

Lemma (Basic Lemma [2])

Let A be a c.e. algebra whose type contains two binary operations $+, \cdot$, and two constants (presented by the numbers) $0, 1$ such that $+$ is associative, the pair of sets $(0_A, 1_A)$ is e.i. (where $0_A = \{x : x =_A 0\}$, and $1_A = \{x : x =_A 1\}$) and, for every a,

$$0 + a =_L a + 0 =_A a, \qquad a \cdot 0 =_A 0, \qquad a \cdot 1 =_A a.$$

Then $=_A$ is a uniformly finitely precomplete (u.f.p.) ceer.

But what does "u.f.p." mean?

Definition (Montagna [10], Shavrukov [16])

An eqrel E is uniformly finitely precomplete (u.f.p.) if it is nontrivial and has a u.f.p. totalizer, i.e. a (total) computable function $f(D, e, x)$ such that

$$\varphi_e(x)\downarrow \;\&\; \varphi_e(x) \in [D]_E \Rightarrow \varphi_e(x)\; E\; f(D, e, x).$$

Examples of u.f.p. ceers: the ceer $\leftrightarrow_T$ corresponding to provable equivalence in any consistent c.e. extension of Q.

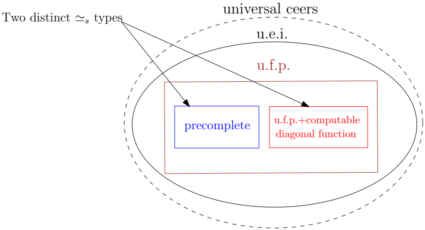
Definition (Russian literature, Ershov [8])

An eqrel E is precomplete if it is nontrivial and has a totalizer, i.e. a (total) computable function $f(e, x)$ such that

$$\varphi_e(x)\downarrow \Rightarrow \varphi_e(x)\; E\; f(e, x).$$

Examples of precomplete ceers (Visser [19]): for evey $n \geq 1$, provable equivalence $\leftrightarrow_{T,n}$ in T restricted to the Σ_n sentences where T is any consistent c.e. extension of $I\Delta_0 + \mathrm{Exp}$.

The world of u.e.i. ceers



Inclusions are proper, except for the open problem: Is u.f.p.=u.e.i.?

A diagonal function for an eqrel E is a function $d : \omega \to \omega$ such that $d(x) \notin [x]_E$ for all x.

Remark

- (Lachlan [9]) All precomplete ceers are strongly computably isomorphic (given eqrels R, S, we say that R is strongly computably isomorphic to S, $R \simeq_s S$, if there is a computable permutation of ω which reduces $R \leq_c S$).
- (Bernardi and Montagna [4]) If R, S are u.f.p. ceers with a computable diagonal function then $R \simeq_s S$. (For instance, $\leftrightarrow_{PA}$, for which a computable diagonal function being is induced by the connective $\neg$.)

To characterize the inclusion of the u.f.p. ceers in the u.e.i. ceers:

Theorem (A characterization of u.e.i. ceers Andrews et al. [1])

A ceer E is u.e.i. iff it has a weakly u.f.p. totalizer, i.e. a u.f.p. totalizer $f(D, e, x)$ which is required to make $\varphi_e(x)$ total modulo E when $\varphi_e(x) \in [D]_E$, only if the elements of D are pairwise non-E-equivalent.

Problem

Is u.e.i. ceers= u.f.p. ceers?

Remark

The usefulness of the Basic Lemma consists in the fact that from effective inseparability of just the pair $(0_A, 1_A)$ one can infer that $=_A$ is not only u.e.i., but even u.f.p.. And ... a lot can be done exploiting u.f.p.-ness.

As a first application, we look at c.e. lattices L, with preordering relation $\leq_L$ and equality $=_L$.

Definition

A c.e. lattice L is said to be effectively inseparable (or simply e.i.) if L is bounded, with, say, the numbers 0 and 1 presenting the least element and the greatest element, respectively, and the pair of sets $(0_L, 1_L)$ is e.i.. Let us also say that a c.e. lattice L is u.e.i. or u.f.p. if so is the ceer $=_L$.

Theorem

If L is an e.i. lattice then L is u.f.p..

Proof. By the Basic Lemma. □

Remark on terminology Henceforth, by any "e.i. structure" we always assume that the structure be. c.e..

Definition

A c.e preorder $\preceq$ is universal, if $R \leq_c \preceq$, for every c.e. preorder R.

Theorem (Universality Theorem [2])

If L is an e.i. lattice then L is universal, i.e. the associated c.e. pre-ordering relation $\leq_L$ is universal.

Proof.

The proof uses the Recursion Theorem and the fact that by the Basic Lemma $=_L$ is u.f.p.. □

Remark

Distributivity is not needed!
But the full lattice structure is necessary. In fact:

Theorem ([2])

There exist an e.i. upper semi-lattice U such $\leq_U$ is not universal.

Definition

If $\preceq$ is a preordering relation (with $\equiv$ its associated equivalence relation), then we say that $\preceq$ is uniformly dense, if there exists a computable function f such that for every a, b if $a \prec b$ then

- $a \prec f(a, b) \prec b$,
- if $a \equiv a'$ and $b \equiv b'$ then $f(a, b) \equiv f(a', b')$.

Theorem (Uniform Density Theorem [2])

If L is an e.i. lattice then L is uniformly dense, i.e. the associated pre-ordering relation $\leq_L$ is uniformly dense.

Proof.

The proof uses the Recursion Theorem and u.f.p.-ness of $=_L$. □

Theorem (Montagna and S, [11] after Pour El and Kripke [14])

If B is an e.i. Boolean algebra then B is universal not only with respect to all c.e. preordering relations (in the sense that $\leq_B$ is universal) but, strengthening universality, it is also universal:

- *w.r.t. all c.e. Boolean algebras, i.e. for each such one B' there is a computable function which induces a monomorphism of Boolean algebras from B' to B (or, rather, from $B'_{=_{B'}}$ to $B_{=_B}$);*
- *and (again in this stronger sense) w.r.t. all c.e. bounded distributive lattices.*

Theorem (Unique strongly computable isomorphism type for e.i. Boolean algebras [14, 11])

All e.i. Boolean algebras are strongly computably isomorphic, i.e. there is a computable permutation of ω yielding an isomorphism of Boolean algebras.

What is lost going from Boolean algebras to lattices:

Remark

We can not expect in general for e.i. (distributive) lattices universality with respect to the full class of c.e. (distributive) lattices.

In fact:

Theorem

There is an e.i. distributive lattice L such that in the corresponding quotient lattice $L_{/=_L}$ one can embed (as a lattice) no infinite Boolean algebra.

And of course we loose uniqueness of isomorphism type (even without requiring computability), and so on.

Corollary (A case in which u.e.i.=u.f.p.)

For any c.e. lattice L, $=_L$ is u.e.i. if and only if $=_L$ is u.f.p..

Proof.

By uniformity of the proofs of the previous results. □

Corollary (Local Universality Theorem)

Any u.e.i. lattice L (in particular any e.i. lattice L) is locally universal, i.e. for every pair $a <_L b$, the preordering relation $\leq_{[a,b]_L}$ is universal.

Proof.

Immediate, as $[a, b]_L$ is an e.i. lattice. □

C.e. Lindenbaum lattices of sentences and uniform density

Definition

A lattice of sentences is a c.e. lattice $L_{\mathcal{C},T}$, where T is a (classical or intuitionistic) formal system of arithmetic; the universe $\mathcal{C}$ is a c.e. set of sentences identified with ω, with operations induced by the propositional connectives $\vee$ and $\wedge$, closed under these operations, with pre-ordering relation $\leq_{L_{\mathcal{C},T}}$ induced by $\rightarrow_T$:
$\alpha \leq_{L_{\mathcal{C},T}} \beta$ if $T \vdash \alpha \rightarrow \beta$,
and
$\alpha <_{L_{\mathcal{C},T}} \beta$ if $T \vdash \alpha \rightarrow \beta$ but $T \nvdash \beta \rightarrow \alpha$;
and equality $=_{L_{\mathcal{C},T}}$ induced by $\leftrightarrow_T$:
$\alpha =_{L_{\mathcal{C},T}} \beta$ if $T \vdash \alpha \leftrightarrow \beta$.

Example (Motivating Example)

T is any classical consistent c.e. extension of Q (or R), and $\mathcal{C} = \Sigma_n$-sentences, for some $n \geq 1$, or $\mathcal{C}$=all sentences.

By the Universality Theorems and the Uniform Density Theorem, one can show that a c.e. lattice of sentences $L_{\mathcal{C},T}$ is locally universal and uniformly dense by simply showing that the pair $(0_{L_{\mathcal{C},T}}, 1_{L_{\mathcal{C},T}})$ is e.i. (where, via coding, $0_{L_{\mathcal{C}}}$ consists of the sentences of $\mathcal{C}$ refuted by T, and $1_{L_{\mathcal{C}}}$ consists of the sentences of $\mathcal{C}$ proved by T).
So it possible to derive, using only computability-theoretic methods, results on density and uniform density relative to well known lattices of sentences.

See the paper by Shavrukov and Visser [17] for a beautiful overview of this topic, and its relevance to proof theory and logic.

In the following survey, we write in red previously unnoticed items.

Reviewing the literature concerning known cases

By work of Shavrukov and Visser [17] and Montagna and S. [11] the lattice $L_{\mathcal{C},T}$ is known already to be locally universal or/and uniformly dense if

1. $L_{\mathcal{C},T}$ is an e.i. Boolean algebra: for instance if T is a consistent c.e. extension of Q, and
   $\mathcal{C} = \Delta_n$, with $n \geq 2$,
   or $\mathcal{C}$=all sentences. We have local universality [11] and uniform density [17].
2. $=_{L_{\mathcal{C},T}}$ is a precomplete ceer: for instance if T is a consistent c.e. extension of $I\Delta_0 + \exp$ and
   $\mathcal{C} = \Sigma_n$ with $n \geq 1$. Uniform density comes from [17]. We are not aware of any recognition of local universality, which easily follows from the Local Universality Theorem.

   (By [11], $L_{\Sigma_n,T}$ is universal with respect to all c.e. distributive lattices (even the bounded ones, if $n \geq 2$.))

3 T consistent c.e. extension of Buss's weak system of arithmetic S_2^1, and $\mathcal{C} = \exists\,\Sigma_1^b$. In this case $L_{\mathcal{C},T}$ is locally universal, and (solving a problem in [17]) uniformly dense.

Remark

Language of S_2^1: language of of Q, plus the *shift right* function $\lfloor \frac{1}{2}x \rfloor$, *length* $|x|$, the *smash* function $x \sharp y$ (intended interpretation: $x \sharp y = 2^{|x| \cdot |y|}$).

Σ_1^b is the smallest class of formulas containing the formulas in which all possibly existing quantifiers are sharply bounded, i.e. bounded by the length of a term, and is closed under sharply bounded quantification, the connectives $\vee, \wedge$ and bounded existential quantification. Then $\exists\Sigma_1^b$ is comprised of the formulas which arise from allowing a single unbounded existential quantifier over a Σ_1^b formula.

Local universality and uniform density (based on our theorems via Basic Lemma)

4 T intuitionistic consistent c.e. extension of iQ and
$\mathcal{C}$ any c.e. set of sentences closed under $\vee$ and $\wedge$, containing $\exists\Delta_0$ or at least the sentences of the form

$$\exists y \neg(\tau_0(x,y) \to \neg\forall z \neg(z < y \wedge \tau_1(x,z))$$

where τ_i is roughly equality of two "polynomials";

for instance $\mathcal{C} = \Phi_n$ with $n \geq 3$, or $\mathcal{C} = \Theta_n$ with $n \geq 2$: these classes refer to Burr's hierarchies of formulas.
In this case $L_{\mathcal{C},T}$ is locally universal and uniformly dense.

$\Phi_0 := \Delta_0$
$\Phi_1 := \Sigma_1$
$\Phi_2 := \Pi_2$
for $n \geq 2$, let Φ_{n+1} be inductively defined by

- $\Phi_n \subseteq \Phi_{n+1}$
- if $\varphi \in \Phi_n, \psi \in \Phi_{n+1}$ then $\varphi \to \psi \in \Phi_{n+1}$
- if $\varphi \in \Phi_{n+1}$ then $(\forall x)\varphi \in \Phi_{n+1}$
- if $\varphi, \psi \in \Phi_{n+1}$ then $\varphi \wedge \psi, \varphi \vee \psi \in \Phi_{n+1}$
- if $\varphi \in \Phi_{n-1}$ then $(\exists x)\varphi \in \Phi_{n+1}$.

$\Theta_0 = \Delta_0$
$\Theta_1 = \Sigma_1$
For $n \geq 1$

- $\Theta_n \subseteq \Theta_{n+1}$
- Θ_{n+1} is closed under $\wedge, \vee, \exists, \forall$
- if $\varphi \in \Theta_n$ and $\psi \in \Theta_{n+1}$ then $\varphi \to \psi \in \Theta_{n+1}$.

Word problems of c.e. algebras

Definition

The word problem of a c.e. algebra A is the ceer $=_A$ presenting equality in A.

Thus in a f.p. group $G = \langle X; R\rangle$ the word problem is the ceer which identifies two terms t_1, t_2 of the free group on X if $t_1 t_2^{-1} \in \mathrm{Ncl}(R)$. (Notice slight difference with the classical notion.)

Research Plan.

Given a ceer E, find c.e. structures A such that E can be "realized" as the word problem of A, meaning one of the following: $E \equiv_c =_A$, or $E \simeq =_A$, or even $E \simeq_s =_A$, (where $\equiv_c$ stands for $\leq_c$ & $\geq_c$, and $\simeq$ is computable isomorphism on ceers, i.e. a reduction whose range intersects all equivalence classes).

Theorem (Delle Rose, San Mauro and S. [7])

Every ceer is $\simeq$ to some c.e. semigroup. But there exist ceers E such that $E \not\equiv_c =_S$ for every f.p. (or even finitely generated c.e.) semigroup S.

Fact

$\leftrightarrow_{PA} \simeq_s =_B$, where B is any e.i. Boolean algebra.

Proof. By the strongly computable isomorphism of all e.i. Boolean algebras; or more simply, by the fact that the two ceers are both u.f.p. with a computable diagonal function, and thus lie in the same strongly computable isomorphism type.

Remark

More examples of c.e. algebras A so that $\leftrightarrow_{PA} \simeq_s =_A$ can be found by building A satisfying the hypotheses of the Basic Lemma, and so that $=_A$ has a computable diagonal function.

For instance:

Theorem ([7])

There is a non commutative c.e. ring R such that $\leftrightarrow_{PA} \simeq_s =_R$.

Proof. *Build a c.e. ring R with desired properties so that $(0_R, 1_R)$ is an e.i. pair of sets.*

Problem

Do there exist f.p. groups G such that $\leftrightarrow_{PA} \simeq_s =_G$?

Theorem (Nies and S. [12])

There exists a f.p. group G such that $=_G$ is u.e.i..

Remark Had we built G so that $=_G$ is u.f.p. then we would be OK. But unfortunately, this is not so, and the Basic Lemma does not apply to c.e. groups, so we do not know if $=_G$ is u.f.p..

By next theorem it would be enough to build a f.p. group G such that $=_G$ u.e.i., and possessing a computable strong diagonal function where a function d is a strong diagonal function for an eqrel E if $d(D) \notin [D]_E$, for every finite set D.

Theorem ([3], unpublished)

If E is a u.e.i. ceer with a computable strong diagonal function then E is u.f.p. (and thus $E \simeq_s \leftrightarrow_{PA}$).

Thanks for your patience and attention

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