MATH 132

Problem Solving: Algebra, Probability, and Statistics

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MATH 132 – Problem Solving: Algebra, Probability, and Statistics
Lecture notes version 1.3 (Fall 2018)

Contents

Chapter 1

Probability

1.1 Flipping Coins: An Introduction to Probability

Consider playing a game where there are two teams, the Brewers and the Cubs. You flip two coins at the same time; if the faces that come up match (i.e. both heads or both tails), then the Brewers get a point. If the faces are different, then the Cubs get a point. I'll come around with coins. Don't actually flip any of them until you get to the second question.

1. Suppose you were to flip the coins ten times. Which team do you think would have the most points? Make an exact prediction for the score (like 10-0, or 7-3). For what reasons did you come up with that prediction?

2. Now actually flip the coins ten times and keep score. Did the outcome match your prediction? Do you think this is a coincidence? Do you think your prediction was a good one? a bad one? and why? Do these results make you want to change your prediction? If so, what is your prediction now?

3. What do you think the "probability" is of the Brewers scoring a point? Of the Cubs? Think carefully about how you got this answer. What reasoning and information is it based on? Try to write down explicitly the definition of probability that you are using to get these answers.

Now for some vocabulary to get us started. Any particular performance of a probability experiment is called a **trial**. In this class, we will assume that a trial or experiment is random, unless otherwise noted. Sometimes we use the words trial and experiment interchangeably, but if you need to distinguish between the two words, consider an experiment to be the larger entity, formed by a number of trials. Each trial results in one or more **outcomes**. For example, we just ran a probability experiment in which a trial was simultaneously flipping two coins. An outcome of one trial was, for example, "two heads."
A basic but very important concept is the idea of a **sample space**, which is a list of *all* the possible outcomes of a trial. Consider our probability experiment of simultaneously flipping two coins. One student might give a sample space of

$$\{\text{two heads},\ \text{one head and one tail},\ \text{two tails}\},$$

while others student might give sample spaces of

$$\{\text{head-head},\ \text{head-tail},\ \text{tail-head},\ \text{tail-tail}\}\ \text{ or even }\ \{\text{no heads}, \text{some heads}\}.$$

Notice that in each list, every possible outcome of a trial is listed. So they are all valid sample spaces. But they are undoubtedly different; they're not even of the same size!
Lastly, we define an **event** to be any subset of a sample space (any collection of outcomes). For example, the subset $\{\text{head-tail},\ \text{tail-head},\ \text{tail-tail}\}$ of the second sample space above is an event. Notice we can say this in English more succinctly as the event *flipping at least one tail*. Think about the difference between outcomes and events. Can outcomes be events?

Having different sample spaces for the same probability experiment is not immediately a problem. The problem comes in with the way many textbooks define probability:

Naïve Definition of Probability: The probability of event A happening is:

$$P(A) = \frac{\text{number of outcomes in event } A}{\text{total number of outcomes in the sample space}}$$

For example, in our probability experiment of simultaneously flipping two coins, let's consider the event "one head and one tail". Going by the first sample space mentioned above, we get $\mathrm{Pr}(\text{one head and one tail}) = \frac{1}{3}$, but going by the second sample space mentioned above, we get $\mathrm{Pr}(\text{one head and one tail}) = \frac{2}{4}$. These are undoubtedly different answers. What happens if we try to calculate this probability using the third sample space?
A simple event like this should have a certain probability of happening, and we should be able to calculate it. Furthermore, the answer we get shouldn't change depending only on how we choose to think about the problem. So...

1. Is the probability of getting one head and one tail actually $\frac{1}{3}$ or actually $\frac{2}{4} = \frac{1}{2}$? If you had a student who wasn't sure, what would you suggest they try to get an idea of which one is correct? Why?

2. Suppose we want to find the probability of getting a spade when randomly drawing a card from a standard 52 card deck (with 13 cards in each of the four suits). Think about the following three sample spaces. For each, what does the naïve definition give as the probability of choosing a spade? *Use the definition of probability given above and the sample spaces, not any prior knowledge!*

   - $\{\text{club, diamond, heart, spade}\}$
   - $\{\text{spade, not a spade}\}$
   - $\{\text{A}\clubsuit, 2\clubsuit, 3\clubsuit, ..., \text{K}\clubsuit, \text{A}\diamondsuit, 2\diamondsuit, ..., \text{K}\spadesuit\}$ (i.e., all 52 cards)

   One of the above is not like the others. Why does it give you a different answer? Be specific.

3. What's wrong with the naïve definition? That is, how *should* you calculate probabilities? This should include some thoughts on how to choose a useful sample space.

Better Definition of Probability: The probability of event A happening is:

1.2 Sample Spaces and Equally Likely Outcomes

1. Two students are playing a game called "Evens and Odds," where they each roll one die and then multiply the numbers that turn up. Brandon gets a point if the product is even, while Melissa gets a point if the product is odd. Before they play, they are asked to come up with a sample space, and then give the probability that each will win a point.

   (a) Brandon answers that there are only two possibilities, one being "Product is Even" and one being "Product is Odd", so that each player is equally likely to win a point. Is this reasoning correct? If not, what is wrong with it?

   (b) Melissa answers that the sample space is this:
   (1; 1) (2; 2) (3; 3) (4; 4) (5; 5) (6; 6)
   (1; 2) (2; 3) (3; 4) (4; 5) (5; 6)
   (1; 3) (2; 4) (3; 5) (4; 6)
   (1; 4) (2; 5) (3; 6)
   (1; 5) (2; 6)
   (1; 6)
   Counting up these possibilities, there are 6 combinations which give an odd product, and 15 which give an even product. Thus, the probability of Player 1 winning is $\frac{15}{21}$, or $\frac{5}{7}$, while the probability of Player 2 winning is $\frac{6}{21}$, or $\frac{2}{7}$. Is this reasoning correct? Why or why not?

   (c) What sample space would you use (one of the above or another)? Why?

2. Suppose the letters of the word MASSACHUSETTS are each written on a separate piece of paper and placed in a paper bag. As one trial, you choose one piece of paper at random from the bag.

   (a) Find a sample space for this scenario.

   (b) Are each of the outcomes in your sample space equally likely? Why or why not? If not, can you come up with a way to write a sample space in which the outcomes are equally likely? If your space already has equally likely outcomes, what different space might you see younger students coming up with first?

   (c) Find the probability of the event *choosing an 'S'*, and of the event *choosing a vowel*.

3. Suppose, as one trial, you draw two coins (without replacing the first one) from a purse which contains two quarters, one nickel, and one dime.

   (a) Find a sample space for this scenario. Are the outcomes equally likely?

   (b) If you have not yet done so, come up with a way to write a sample space in which the outcomes are equally likely.

   (c) Find the probability of the event *choosing at least one quarter*.

4. (a) Consider our Brewers vs Cubs coin game from the first day of class. Suppose we want to predict what the score will be after 3 rounds of flipping coins; for instance, say we want to find the probability that the Brewers are ahead after 3 rounds. We originally gave a sample space, with equally likely outcomes, of
{head-head, head-tail, tail-head, tail-tail}. Can we calculate the probability that we want using only our definition and this sample space?

   (b) What has changed? What should be considered as a trial? What are the outcomes of these trials (i.e., what is the sample space)? Are these outcomes equally likely? If not, adjust your sample space so that they are.

   (c) Now calculate P(the Brewers are ahead after 3 rounds).

In summary: when we want to calculate a probability from our definition, we need to create a sample space that will work for us. First of all, we need that all outcomes are equally likely. Also, recall our definition of probability– we can only calculate probabilities of *events*, and what counts as an event depends on what we choose as a *sample space*. Briefly discuss a strategy for approaching a probability calculation like that from part (a) above.

1.3 Modeling Probabilities

We have seen that sometimes it is very difficult, or seems like unnecessary work, to come up with a sample space of equally likely outcomes that allows us to calculate the probabilities of the events we care about. A sample space of equally likely outcomes is crucial to the definition of probability (remember the definition!), but maybe we can find a way to work with probability so that the sample space that we need is relegated to the background.

As in many areas of math (think back to 130 and 131), pictures can do wonders in terms of understanding and justification. We will work with two models: area models and tree models.

In the aptly named **area model**, probabilities are represented by areas. Think about the picture below as representing the probabilities of the possible outcomes of rolling a fair die.

1
2
$\rightarrow$ 3 $\leftarrow$
4
5
6

We think of the whole box as having area 1. We split the big box into six pieces and label them, one for each possible outcome of the die. Since all faces are equally likely, we make all the pieces the same size; that is, since the outcomes have the same probability, we make the corresponding pieces have the same area. Now we can see that the box for 3 is $\frac{1}{6}$ of the total area, so P(rolling a 3) $= \frac{1}{6}$.

You could also think of the big box as a dartboard - if you randomly threw a dart at it, (assuming you hit the board!) you'd have a 1 in 6 chance of hitting the box with the 3 in it.

Recall this problem: you draw two coins (without replacing the first one) from a purse which contains two quarters, one nickel, and one dime. Find the probability of choosing at least one quarter.

Let's construct a corresponding area model. The first thing that happens is that we pick one coin out of the four. As with the die example above, we'd draw a box like the one at left below. In our last solution to this problem, we realized that we have to treat the quarters separately. But note that we might as well have taken that into account by drawing the model like the one at right. This will make our lives easier.

Q
Q
N
D

<table>
<tr><td rowspan="2">Q</td></tr>
<tr></tr>
<tr><td>N</td></tr>
<tr><td>D</td></tr>
</table>

Now we need to figure out what to do about the second coin. We can do that as follows. Suppose that the first coin we picked was the dime. Then we know that our "dart" has landed somewhere in the bottom box of the model. So we "zoom in" this box and repeat. For the second coin, we could get either of the two quarters, or the nickel. Since there are three coins, we think of this bottom box as split into three equal pieces. One is for the nickel, and, as above, we draw the two for the quarters together. Then we suppose we had drawn the nickel as the first coin, and fill in the boxes accordingly. Lastly suppose we had drawn a quarter first, and fill in that box too.

1. We are asked to find P(choosing at least one quarter). We have a bunch of little pieces of our model. Which boxes do we care about? Find the box that represents picking both quarters. What is the area of this box? How do you know? Then what is P(draw two quarters)? What is P(choosing at least one quarter)? Why?

2. Suppose we had treated the two quarters differently. Then the model would be:

   Now look back at the sample space we found last class to solve this problem. What do you notice?

We can also use a **tree** to model probability. For each "thing" that happens (we are careful not to use the word event), we start with a point, and draw a line from that point for each different outcome that can happen. If another activity then occurs, we create new tree models starting from the end of each of the previous branches.

3. I have a strange but fair four-sided die, with sides labeled 1,2,3,4. Draw a tree model for rolling this die twice, and find P(roll two 1's). Think ahead to be neat– how large will this tree be? Also find P(roll a 1 and a 4, in either order).

Now, suppose sides 1, 2, and 3 are painted blue; side 4 is red. I will ask you for P(roll blue twice). But instead of using the above model, note that, now, numbers don't matter, only color, so that sides 1, 2, and 3 are essentially the same. Then maybe we can represent them by one branch instead of three (we did something similar in the area model). But now, the probability of blue and red are not equally likely, so we better label this model with the correct probabilities.

4. Finish the rest of the model like this. Now find P(roll blue twice) using this model. What should you do? Why? Use the model from problem 1 for ideas, and justify your calculation, again as if to a 7th grader.

5. We always need the sum of the probabilities of all the outcomes to be 1 (why?). In the area model, we decree that the area of the whole box is 1, so that is is trivially true. How can we be sure this happens in the tree model?

Remember: the only definition of probability we have relies on a sample space of equally likely outcomes. These models use that definition, and in fact, we've seen that if we draw the models always using equally likely outcomes, the sample space shows up. However, the models allow us to skip the "equally likely" step as long as we label and think correctly. That's a nice improvement.

6. Previously you thought about a strategy for probability calculations using sample spaces. Come up with a new strategy that uses these models, including thoughts on how to decide which model may be best to use.

1.4 More Modeling

1. Suzie got two A's this week, and so she gets to choose a marble out of a bag of 5 marbles, 2 of which are red; the rest are white. If she gets a white marble, she is out of luck. But if she gets a red marble, she gets to choose from a bag with equal amounts of cherry, grape, and apple lollipops. Model this situation using a tree diagram, and find the probability that Suzie ends up with an apple lollipop.

2. Now think about how you could write down a sample space for this scenario. You don't need to actually come up with one: instead, I want you to focus on why it is difficult to make the outcomes equally likely. Looking at your tree may give you some ideas.

3. Annette plays ultimate frisbee, and sometimes her games have to be canceled because of the weather. During her season, there is a 4/5 chance that it is cold; the other 1/5 of the time it is warm. Also, regardless of temperature, there is a 25% chance of precipitation. Annette's games are played if it is warm or there is no precipitation.

   (a) Draw a tree diagram representing both temperature and precipitation.

   (b) What is the probability that Annette plays her game? In other words, find P(*it is warm or there is no precipitation (or both)*), from your tree.

   (c) Why is this not equal to P(*it is warm*)+P(*there is no precipitation*)?

   (d) Find a way to calculate P(*it is warm or there is no precipitation (or both)*) using P(*it is warm*) and P(*there is no precipitation*) (and maybe some other probabilities).

4. In basketball, a free throw is worth one point. Suppose Yao gets fouled in the act of shooting, so he will get two free throw shots. Create both an area and tree model, and find the probability that he gets 0 points, 1 point, and 2 points in each of the following scenarios:

   (a) Each of Yao's shots has a 60% chance of going in.

   (b) Now suppose that, given Yao makes the first shot, he'll feel more confident and so on his second attempt he will make the shot with 80% probability. Given that he misses, he will feel less confident and his second attempt will have only a 50% probability of going in.

   (c) There is a difference between your two tree diagrams. What is it, and what caused it?

1.5 Expected Value

1. A parking lot charges \$7 to park per day. A car that is illegally parked will receive a \$25 ticket if caught, and the chance of being caught is 60%. On average, is it cheaper to pay to park or to risk getting a ticket? Can you figure out a way to quantify your answer?

2. Now suppose you learn that Jack, one of the two parking enforcers, is very nice, so that if he catches your car parked illegally he will just charge you the \$7 you should have paid. If Jill, the other enforcer, catches your car parked illegally, however, she'll still charge you the \$25 ticket. Suppose the chance of a car being caught by Jack is 30%, and the chance of being caught by Jill is 30% (so the change of being caught at all is still 60%). Should you change your answer from above? Why?

The value we found for each situation is called the **expected value** of that situation. You can find the expected value for any probability experiment that has numerical outcomes (here it was an amount of money paid - it doesn't really make sense to find expected value when the outcomes are grape, cherry and apple lollipops).

Intuitively, the expected value tells you the average value of the outcomes if you were to repeat the experiment many times.

Mathematically, we find the expected value by adding up the values of each outcome times their probabilities. This is really just a weighted average.

Consider the following two scratch ticket games. (A payoff of 0 with probability 4 in 5 means that if you buy an Easy Money ticket, it has a 4/5 probability of being worth 0, for example. A ticket has only one monetary value.)

Easy Money

Payoff	Probability
\$0	4 in 5
\$1	3 in 20
\$5	1 in 20

Big Bucks

Payoff	Probability
\$0	999 in 1000
\$1000	1 in 1000

1. Without doing any calculations, which game would you rather play? Why?

2. Ignoring for a moment how much each ticket costs, determine which game is a better investment. That is, if you could play each game many, many times, which game would win you the most money? (How does expected value help you here?)

3. Determine how much the lottery commission would have to charge for each ticket in order to break even in the long run.

4. What sort of reasoning or intuition are you using in answering number 3? Try to be specific.

5. While working on this above, Thom announces that the expected value for the Easy Money game is \$0. When asked why, he says, "you'd *expect* to win that prize the most often. If you played a whole bunch of times, you'd get \$0 more often than any other prize. I mean, *on average* you'd get \$0." How would you respond to Thom's reasoning?

1.6 Using Models in Probability

1. A certain slot machine has 3 windows. In each window, you can get either a cherry or a lemon. In each window, a lemon is twice as likely to come up as a cherry. If you pull and get all cherries, you win $10. If you pull and get all lemons, you win the booby prize of $1. Determine the amount of money you should charge per pull so that this machine breaks even in the long run. Draw a tree model to help you find the probabilities you need. (Hint: you can treat each window as a trial.)

Expected value can show up in situations that don't involve money. We can talk about expected value for any probability experiment which has numerical outcomes.

2. Jay is betting his marbles in a game with his friend Caleb. Each turn, first Jay will flip a fair coin. If the coin comes up heads, Jay wins a marble from Caleb and the round is over. Otherwise, Jay flips the coin again. If this second coin is tails, Caleb wins four marbles from Jay, but if it's heads, they call the round a tie.

   (a) Use an area model to find Jay's expected marble winnings per turn. What does this number tell you about Jay's prospects in this game? Why does this make sense?

   (b) Suppose the boys ask you to make the game fair. What adjustments could you make to do this?

3. We've talked about expected value being a *weighted average*. How can you see this using these models?

4. Annette enjoyed ultimate frisbee so much, she decided to play in the summer, too. However, sometimes her games still have to be canceled because of bad weather. During the summer, 3 out of 4 days are warm; the rest the time it is hot. If it's warm, there is a 10% chance of storms. In hot weather, the chance of storms is 40%. Unless it is storming, Annette will play her game.

   (a) Draw an area model representing the summer weather.

   (b) What is the probability that Annette's game is canceled?

Chapter 2

Statistics

2.1 Measures of Center

We've discussed some ways in which two quantities can be related, for example a proportional or linear relationship, and we've also practiced finding patterns in sequences of numbers. In real life, however such relationships are usually not so obvious, or so nicely defined. Instead, we often have only a collection of data points, and try to describe a pattern or relationship as best we can. In the next few weeks, we will be discussing ways to analyze, describe, and represent sets of data. Consider these three sets of data points, taken from students' scores on an algebra quiz (out of 10 points):

Class A: $\{4, 7, 4, 3, 4, 1, 6, 10, 4, 6\}$
Class B: $\{9, 4, 3, 2, 7, 10, 3, 6, 5, 9\}$
Class C: $\{6, 6, 7, 1, 7, 6, 5, 6, 7, 8\}$

1. **Don't do any calculations yet**; just look at the data sets. What similarities do they have? What differences? Do you think one class did "better" on the quiz than another class? Why?

2. How might you represent these scores visually to demonstrate your answer from question 1? Come up with at least 2 ways to visually represent the scores. What are the advantages and disadvantages of each of the ways? Suppose instead of the scores being out of 10, the scores were out of 100. What would be the advantages and disadvantages of your methods?

Probably the most common way to analyze sets of data is to find the **average**. There are typically three different definitions of average: **mean, median, and mode.** These three values describe the **center** of a data set, in various senses.

> The **mean** of a data set is the sum of all the data points divided by the total number of data points.
>
> The **median** of a data set is the middle number of the data set when it is placed in order. If there is an even number of data points, then we typically take the mean of the two middle points.
>
> The **mode** of a data set is the number that comes up most often.
>
> There can be more than one mode for a data set: you can have two or more data points which all show up the same number of times. You would list all of these numbers as the mode.

3. Calculate the mean, median, and mode for each of the data sets. Do any of these values match up with your reasoning in question 1?

4. What exactly does the mean tell you about a data set? Don't use the word "average" in your answer.

   For example, if you had a student ask you "Why do we care about the mean?", how would you answer the student? Can you come up with a visual way of explaining the mean?

5. What exactly does the median tell you about a data set?

6. The mean and median both try to somehow describe the “center” of a data set. What can you say about a data set where the mean is much larger than the median? Much smaller? (Trying to construct such data sets might be helpful.) Discuss the strengths and weaknesses of the mean and the median. Why might you choose to use one over the other?

7. What does the mode tell you about each data set? Can you come up with an example where the mode tells you something that the mean and median don’t?

8. The data set of daily high temperatures in Madison for the month of May last year has a median of 70, a mean of 66, and a mode of 60. What sorts of conclusions can you make from this data? What sorts of conclusions can you *not* make, i.e., what *doesn’t* this data tell you?

2.2 Measures of Spread

Sometimes we are not only interested in the center of a data set, but also the **spread** of a data set, which in a sense describes whether our points are all clumped close together, or if they are more spread out. Quantities that are commonly used in statistics to help describe this are **range**, **MAD**, and the **standard deviation**[1], and also the idea of **outliers**. The **mean absolute deviation (MAD)**, also referred to as the "mean deviation" is the mean of the data's absolute deviations around the data's mean: the average (absolute) distance from the mean. MAD has been proposed to be used in place of standard deviation since it corresponds better to real life.Because the MAD is a simpler measure of variability than the standard deviation, it can be useful in school teaching.

1. The **range** of a data set is defined as the highest point minus the lowest point. Find the range of each set here. Also, would you consider any the data points **outliers**, which are points that are far away from the rest of the set?

   To find the **mean absolute deviation** of the data, start by finding the mean of the data set.
   Find the sum of the data values, and divide the sum by the number of data values.
   Find the absolute value of the difference between each data value and the mean:
   $|\text{data value} - \text{mean}|$.
   Find the sum of the absolute values of the differences.
   Divide the sum of the absolute values of the differences by the number of data values.

   Here's how we **calculate a standard deviation**:

   First, we subtract the mean from each of the data points.

   Second, we square each of these new values.

   Third, we take the average of the squared values.

   Finally, we take the square root of this average.

[1] The standard deviation is closely related to a quantity you may have also heard of called the **variance**; the variance of a data set is the square of the standard deviation. You may also see a slightly different formula for standard deviation if you take a statistics class, but the idea is the same.

2. Calculate the MAD and standard deviation of $\{1,3,4,5,5,5,7,10\}$.

3. What is the MAD and standard deviation measuring? Find the MAD and standard deviation of $\{4,4,4,4,4,4,4,4\}$, and of $\{2,2,2,2,10,10,10,10\}$ (hint: think before you calculate).

4. What is the MAD and standard deviation of $\{8,8,8,8,16,16,16,16\}$? Compare these to your calculations for $\{2,2,2,2,10,10,10,10\}$. Why does this make sense?

5. In statistics, **outliers** are often defined as points that are more than 2.5 standard deviations away from the mean. Why is this a reasonable way to define an outlier?

6. Which class's data set do you think has the bigger MAD or standard deviation? Explain why, and what the number represents in this context. (Don't try to calculate it exactly, just think.)

2.3 Histograms, Stem-and-Leaf Plots, & Box Plots

Consider the following ordered data set of exam scores (out of 100):

$$\{39, 43, 47, 52, 52, 54, 58, 59, 60, 67, 73, 74, 76, 77, 78, 82, 89, 93, 93, 96, 97\}$$

Would a pie graph be appropriate? What about a bar graph or line graph? What would your bars count?

When we use a bar graph to count the number of data points that fall into a *range* of values, we call it a **histogram**. Create a histogram for our data.

We can also create a **stem-and-leaf** plot by first ordering the data points (here they are already ordered), and then grouping them by their first digit in a table. Draw a stem-and-leaf plot for the data here.

How is this like the histogram? How is it like a frequency table? Is it more helpful than these other two types of graph? What if our data points were scores out of 1000? What if they were heights, like 5'3"? Discuss strengths and weaknesses of these two methods.

Yet another way to represent this data is the **box-and-whisker plot** (sometimes called a **boxplot**). This is a relatively new graph (it was first invented by John Tukey in 1977), but it has gained popularity because it has many useful properties. We first find a **5 number summary**.

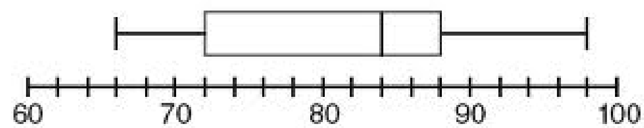
1. First, find the smallest and largest data points of your set.
2. Find the median of your data set. The median is sometimes called the **middle quartile**.
3. As we've discussed, the median divides your data into 2 halves (if your data set has an odd number of points, don't put the median into either of the halves). Find the median of the bottom half of data points. This is called the **lower quartile**.
4. Find the median of the upper half of data points. This median is called the **upper quartile**.

To draw the box plot, indicate these five numbers on the number line and draw a box spanning from the lower quartile to the upper quartile. Draw a vertical line in your box to indicate the median. Draw a line from your box at the upper quartile to the largest data point, then one from the lower quartile to your lowest data point.

Find the 5 number summary and draw a box-and-whisker plot for our data (copied below). See the next page for an example.

$$\{39, 43, 47, 52, 52, 54, 58, 59, 60, 67, 73, 74, 76, 77, 78, 82, 89, 93, 93, 96, 97\}$$

Now look at the box-and-whisker plot below, which graphs the scores of a different class on the same exam. What does the plot tell you about the data? What differences between class performance do the plots suggest? Can we tell how many students there are in this class?



What are some advantages and disadvantages to graphing with the box and whisker plot?

Sometimes, in box and whisker plots, outliers are plotted as single points (or sometimes asterisks), and then a regular box and whisker plot is found without using the outliers. Why might that be a useful technique?

We've now seen pie graphs, line graphs/bar graphs, histograms, stem and leaf plots, and box and whisker plots. We've also discussed three measures of center and two (range and standard deviation) of spread.

It's all well and good to be able to draw these graphs and compute these statistics. The key is the ability to use them to succinctly explain data, so it's important to be able to reason about which graph and/or statistics you should present in a given situation.

For each of these five graph types, briefly note if and how the graph displays each of the statistics above. You might want to organize your thoughts using a table.

2.4 Scatterplots

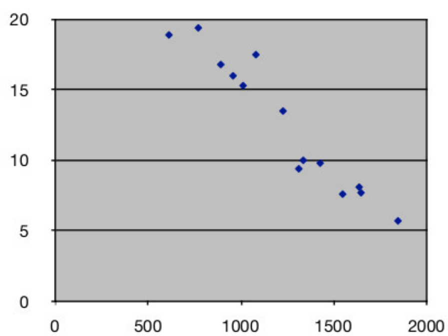
x	611	769	890	956	1010	1079	1225	1310	1334	1425	1546	1635	1645	1844
y	18.9	19.4	16.8	16	15.3	17.5	13.5	9.4	10	9.8	7.6	8.1	7.7	5.7

1. Take a look at the data you are given. What is different about this data from the sets we've looked at previously?

2. I claim that none of graphs that we've talked about so far would give a good pictorial representation of this data. Why do you think?

3. Similarly, I claim that our measures of center and of spread aren't very helpful here either. Why? How would we calculate these numbers?

4. Since we're given the data in x, y pairs, it makes sense to plot each data pair as an (x, y)-coordinate point. This is called a **scatterplot** or **scattergraph**.



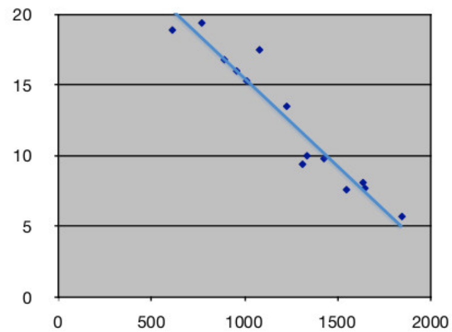
5. What trend or pattern do you notice?

6. Suppose I asked you to predict the y-value for $x = 1500$ using this scatterplot. What would you guess, and why did you guess it? Which data points are you using to come up with your estimate?

7. A computer (or you, with a formula and lots of time) can find what is called the **best-fit line**. This is a line that most accurately fits the data, in the sense that it somehow minimizes the total distance from the line to all of the data points[2]. Draw on your graph your guess of the best-fit line.

[2] The most common choice of a best fit line is called the "least squares regression line."

8. Below, I've used a computer to plot the actual best-fit line. How does it compare to your guess? Use the line to predict the y-value for $x = 1500$.

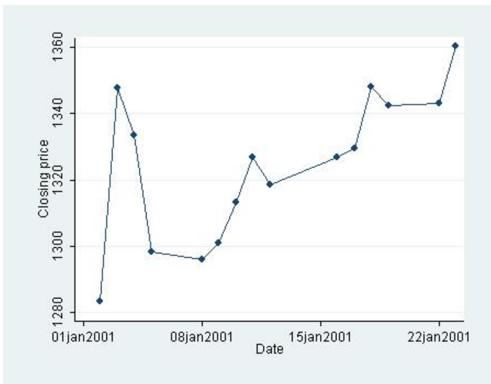


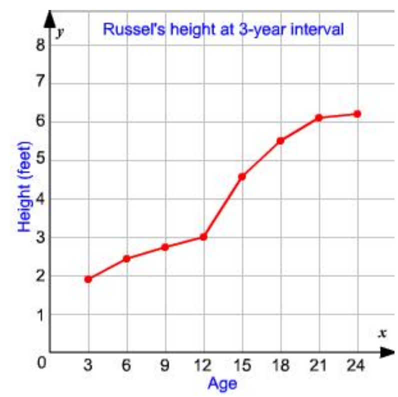
9. Is it close to your estimate from the previous page, when you didn't have the line to help you? Why or why not? Which data points are used in this estimate?

10. Think about one of our first data examples, the quiz scores $\{4, 8, 4, 3, 4, 0, 6, 10, 4, 6\}$. We could plot the points (score, number of students who got that score) as (x, y) pairs in a scatterplot. Think about what this graph would look like, or draw a little sketch if you like. What do you notice?

In your graph, you found a relationship between the points x and y. We're not really trying to find a relationship between the quiz scores above and the number of students who scored them; when we graphed this data we just wanted a visual way to study it. So, we really only want to use a scatterplot when we have **two related sets** (i.e., **pairs** of data) and we want to try to **find a relationship or trend** between them. Otherwise, we should use one of our other types of graph.

Now take a look at the following two graphs. The points were plotted similarly to our scatterplot. In the first the data comes in pairs giving both the date and the closing price of a certain market. In the second, the points are pairs giving Russel's height and his age. This kind of graph is (unfortunately) also usually called a **line graph**.





11. Unlike our scatterplot, here, the points were connected. Why do you think this was done? Why do you think this kind of graph was chosen, rather than a bar graph?

12. What do these line segments mean? What can we say, for example, about the closing market price on Jan 4, or about Russel's height at age 13?

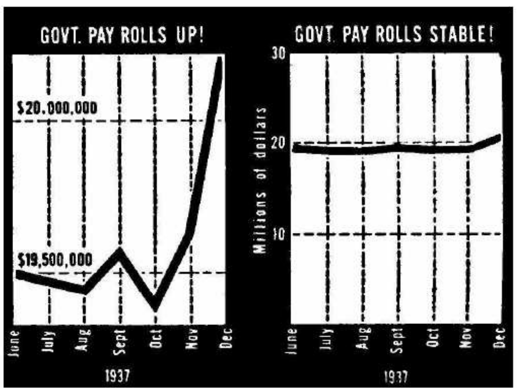
13. Why do you think we did not connect the points in our scatterplots as in these line plots?

14. If you are given data as pairs or numbers and plot them, how should you decide whether to "connect the dots" like this, or whether you should instead draw a best fit line? Could you do both?

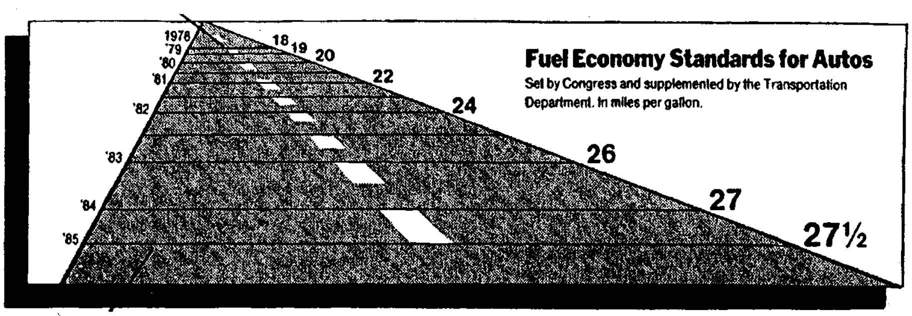
2.5 Misleading Graphs & Statistics

We've talked about how graphs can tell a story. Unfortunately, that means that graphs, unintentionally or otherwise, can tell misleading stories.
Similarly, we use statistics to succinctly describe important features of a potentially large data set so that it can be analyzed. By their very nature, then, statistics often don't tell the whole story. Not to mention, statistics are often calculated or interpreted completely incorrectly. A quote popularized by Mark Twain speaks of "lies, damned lies, and statistics." It's important to be able to look with a critical eye.

1. Below are two graphs which show how the government payroll changed in the second half of 1937. Does the information in one contradict the information in the other? Why or why not? Do you find one or both graphs misleading, and if so, how? How much did the payroll change?

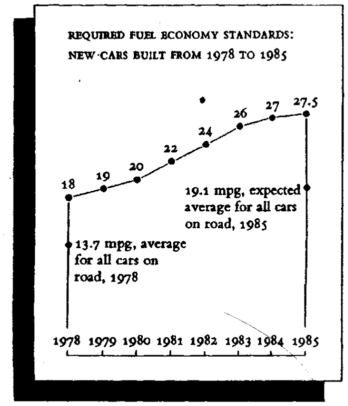


2. What are your first impressions from the following graph? How would you describe the increase in mpg standards?



Now look more closely at the numbers. How is this graph misleading?

3. The graph at right represents the same data. Do you think it is a "more honest" representation? Why or why not?



4. During the Spanish-American War, the death rate in the Navy was nine per thousand. For civilians in New York City during the same period it was sixteen per thousand. Navy recruiters later used these figures to show that it was safer to be in the Navy than out of it. What do you think?

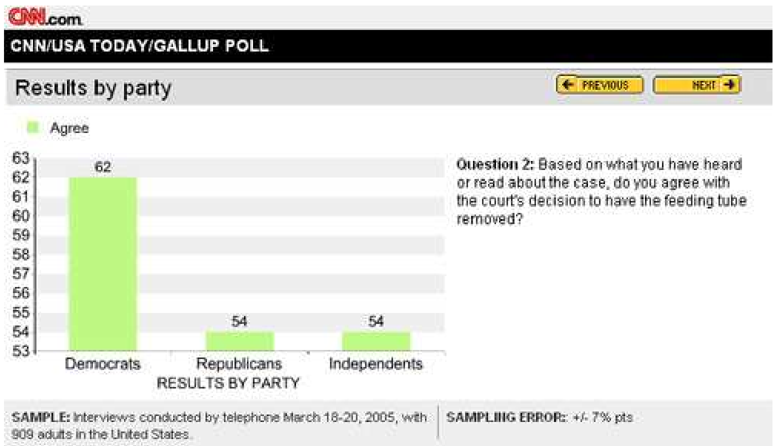
5. Is there anything fishy about the following newspaper excerpt?:

   > In the governor's new budget, state workers were given a 5% increase in wages. This is a hollow victory considering the 20% pay cut these same workers got in the last budget. It's hard to celebrate what amounts to a 15% pay cut.

6. A human can contract malaria from a single bite from a malaria-infected mosquito. Paul is taking a trip to a tropical region and is trying to decide what precautions to take. He reads a traveler's report stating that 1% of the region's mosquitos are infected. Paul's Dad, who is very worried about Paul's trip, exclaims that "after just 100 bites, you'll have malaria!"

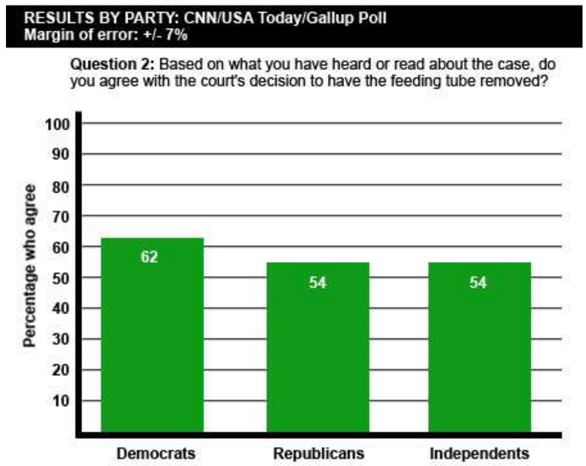
   Suppose we assume that Paul will definitely get malaria if he gets a bite from an infected mosquito. What do you think of his Dad's conclusion?

7. in 2005, CNN polled 909 adults via telephone and asked if they agreed with the court's decision to have Terri Schiavo's feeding tube removed. The results were posted on CNN.com in this manner:



What are your first impressions? What conclusions did you draw? How is this graph misleading?

After complaints, CNN replaced it with a graph that looked like the following:



Why is this graph better?

Beyond the graphical representation, do you notice other issues with this data? E.g., How was it collected? What error do they report?

Chapter 3

Algebra

3.1 Patterns and Rules

Find numbers that fit into the patterns below. Then try to write a general rule for the patterns. If you can think of multiple ways to state your rule, great – write them all down. Be as clear as possible. You should be able to explain what each part of your rule means, and how the part is related to the pattern you are observing.
If you get really stuck on one or two of these patterns, move on. We'll come back to them later.

1. 3, 6, 9, ___, 15, ___, ...
2. 1, 4, 7, ___, 13, ___, ...
3. 1, 2, 4, ___, 16, ___, ...
4. 1, 4, ___, 16, 25, ___, ...

Let's look again at the first pattern, presented in a different way. Since the 1st number is 3, the 2nd is 6, etc., it makes sense to write

1	3
2	6
3	9
4	12
5	15
6	18

Does this representation suggest any different rules? Also, draw a graph in the space above, with the left column as x-values and the right column as y-values. What rules or patterns does the graph suggest?

For younger students, **tables** and **graphs** are great ways to to explain rules, and to display a pattern in an organized way. It's also important to be able to explain your rules **in words** (like: "you start with 3, and then to get the next number, you add 3 to the current number").

We also, however, want to be able to write rules **algebraically**.

For example, the expression $3 \times n$, or $n \times 3$, or $3n$ can be used to describe the pattern above. But it's important to think about what we mean.

1. What is n? Where can you "see" n in the pattern, the table, and the graph? What numbers is n allowed to stand for?

2. Write down algebraic expressions for the following patterns. First writing them in table form might help.

   (a) 1, 4, 7, ___, 13, ___, ...
   (b) 1, 2, 4, ___, 16, ___, ...
   (c) 1, 4, ___, 16, 25, ___, ...

We described the following pattern using the algebraic expression $3 \times n$, $n \times 3$, or $3n$

$$3, 6, 9, 12, 15, 18, ...$$

Another perfectly valid rule for this pattern is to say "you start with 3, and then to get the next number, you add 3 to the current number." How can we express this algebraically? We'll use the variable n just as we did before, but now I need some way to algebraically write down "the current number" and "the next number," keeping in that as I fill in the pattern "the current number" changes.
We will use the notation a_n to mean the number at the nth spot in the pattern. So, for example,

$$a_1 = 3,\ a_2 = 6,\ a_3 = 9,\ a_4 = 12, ...$$

We chose the letter a, but any letter will do as long as we always remember to define our variables.

Now we can write "you start with 3, and then to get the next number, you add 3 to the current number" as

$$\begin{aligned} a_1 &= 3 \\ a_2 &= a_1 + 3 \\ a_3 &= a_2 + 3 \\ a_4 &= a_3 + 3 \\ &\ldots \end{aligned}$$

But there's a pattern here. All of these statements can be summarized in the two statements

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = a_n + 3$$

The first part just tells us that our first number is 3. The second part tells us that to get to the $(n+1)$th number, we take the nth number and add 3. This is all we need to recreate the whole pattern! Since we're given the first number a_1, to find the second number a_2, we just have to pattern match. We have a rule that tells us how to find a_{n+1}. So if we want a_2, we should pick n to be 1, so that

$$\begin{aligned} a_{n+1} = a_n + 3 \;\rightarrow\; a_{1+1} &= a_1 + 3 \\ a_2 &= 3 + 3 \;\; (\text{remember } a_1 = 3) \\ a_2 &= 6 \end{aligned}$$

Be aware: there is arithmetic happening in the usual places, but there is also arithmetic happening in subscripts! Point out where this happened.
So, now we have the second number in the pattern. Find the third number a_3 in the pattern the same way as above, except, since we now know the second number, we should substitute $n = 2$.
Now find the fourth number a_4:
These algebraic equations for a_{n+1} and a_n, together with a_1 is called a **recursive form**:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = a_n + 3$$

We can also use our algebraic expression $3 \times n$ to get a form,

$$a_n = 3n$$

This type of equation is called a **closed form** or **explicit form**. Describe the difference between these two kinds of forms. Any idea why we use those words to characterize them?

1. Write closed and recursive forms for describing these patterns.

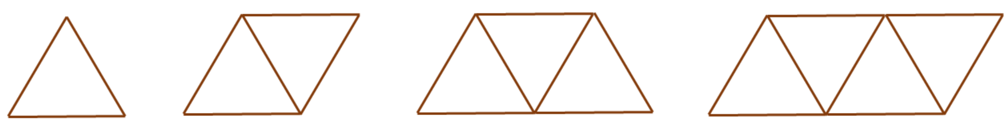
   (a) 3, 6, 9, ___, 15, ___, ...

   (b) 1, 4, 7, ___, 13, ___, ...

   (c) 1, 2, 4, ___, 16, ___, ...

   (d) 1, 4, ___, 16, 25, ___, ...

2. Study the chains of triangles made with toothpicks in the figure below and answer the questions that follow.



   (a) Describe the shapes. Is there a pattern? Can you describe the pattern?

   (b) Make next three shapes.

   (c) How many toothpicks are needed for the first, the second, and the third shapes?

(d) Is there a quick way to find out the number of toothpicks needed for the 10th shape? Is there a pattern to help us out?

(e) Describe how the pattern grows. What about the 100th shape? Is there a way to tell easily the number of toothpicks needed for a shape?

(f) Can this way be written in symbols?

3. Consider the arithmetic sequence whose first few entries are 6, 11, 16, 21, 26, 31, ...

(a) Determine the 100th entry in the sequence and explain why your answer is correct.

(b) Find an explicit form for nth entry in this sequence and explain in detail why your rule is valid.

(c) Find a recursive form that describe this sequence.

(d) Is 1000 an entry in the sequence or not? If yes, which entry? If no, why not? Determine the answer to these questions in two ways: with algebra and in a way that a third, fourth, or fifth grader might be able to.

(e) Is 201 an entry in the sequence or not? If yes, which entry? If no, why not? Determine the answer to these questions in two ways: with algebra and in a way that a third, fourth, or fifth grader might be able to.

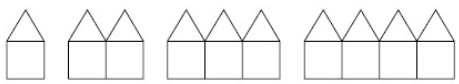
Arithmetic Sequences - If the terms of a sequence differ by a constant (common difference), we say the sequence is arithmetic. If the initial term a_1 of the sequence is a and the common difference is d, then we have, a recursive form $a_n = a_{n-1} + d$, $a_1 = a$ and a closed form $a_n = a + (n-1)d$ for describing such a sequence. Check what other (equivalent) equations can describe the same sequence.

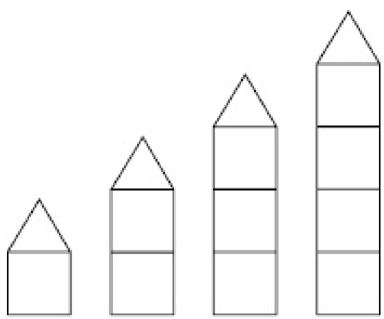
4. Consider an arithmetic sequence whose third entry is 10 and whose fifth entry is 16. Use the most elementary reasoning you can apply to find the first and second entries of the sequence. Find an explicit form for nth entry. Find a recursive form to describe this sequence.

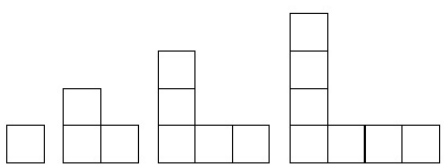
Geometric Sequences - A sequence is called geometric if the ratio (common ratio) between successive terms is constant. Suppose the initial term a_1 is a and the common ratio is r. Then we have, a recursive form $a_n = r \cdot a_{n-1} + d$, $a_1 = a$ and a closed form $a_n = ar^{n-1}$ for describing such a sequence. Check what other (equivalent) equations can describe the same sequence.

5. Consider the geometric sequence whose first few entries are 2, 10, 50, 250, 1250, 6250,... Find the explicit form for nth entry in this sequence and explain in detail why your rule is valid.

6. For each of the following patterns, find two different (two explicit, two recursive, or one explicit/one recursive) formulas to compute the perimeter of the nth figure. **Justify each formula using the figures.** Do NOT just give the formulas.

   1. 

   2. 

   3. 

7. (challenge) Fibonacci sequence was first observed by the Italian mathematician Leonardo Fibonacci in 1202. He was investigating how fast rabbits could breed under ideal circumstances. He made the following assumptions:

   (a) Begin with one male and one female rabbit. Rabbits can mate at the age of one month, so by the end of the second month, each female can produce another pair of rabbits.

   (b) The rabbits never die.

   (c) The female produces one male and one female every month.

   Work with your neighbor to see if you can develop the sequence themselves. Remember that youre counting pairs of rabbits (the number in parentheses), not individuals.

   (a) Begin with one pair of rabbits.

   (b) At the end of the first month, still only one pair exists.

   (c) At the end of the second month, the female has produced a second pair, so two pairs exist.

   (d) At the end of the third month, the original female has produced another pair, and now three pairs exist.

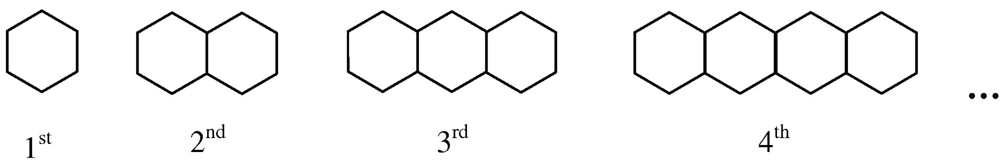
   (e) At the end of the fourth month, the original female has produced yet another pair, and the female born two months earlier has produced her first pair, making a total of five pairs.

   Write the pattern and discuss the sequence.

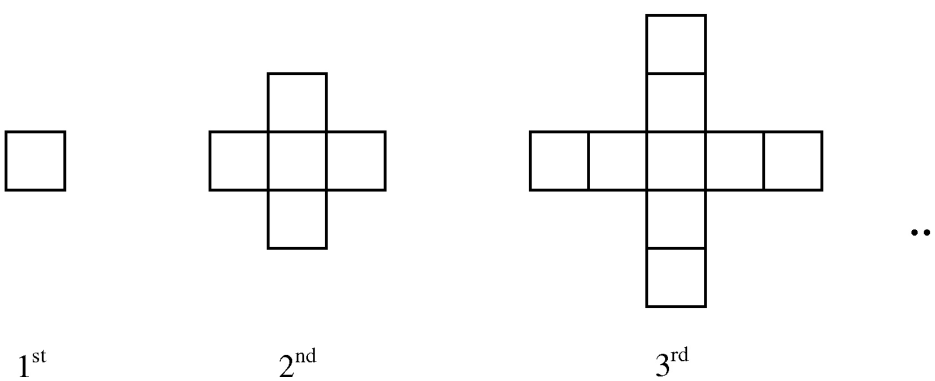
3.2 Patterns and Rules II

Writing rules means write algebraic equations that describe the rules. You can use either explicit forms or recursive forms.

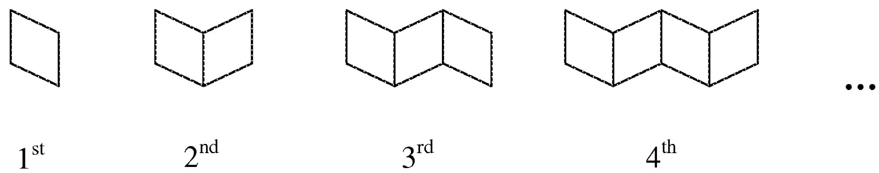
1. Write at least two different rules (two explicit, two recursive, or one explicit/one recursive) for the **perimeter** of the figure in the nth position in the pattern below (all the sides have length 1).



2. Write at least two different rules for the **number of blocks** in the figure in the nth position in the pattern below.



3. Write at least two different rules for the **number of edges** of the figure in the nth position in the pattern below.



4. Find numbers that fit into the tables and sequences below, and then find at least one rule.

(a) ... ___, -9, -2, 5, ___, ___, ...

(b) ... ___, 8, 4, 2, ___, ___, ...

(c)

0	3
5	4
10	___
___	6

(d)

-2	4
-1	1
0	0
1	___
2	4
3	___

5. Compare and contrast the types of patterns we've seen so far: sequences, figures, tables, etc. How does the type of pattern change how we think about the problem? How does it change how we write rules for the pattern?

3.3 Linear Relationships

Identifying Linear Relationships

Arithmetic sequences are closely related with linear relationships. In fact, an explicit form, $a_n = a + (n-1)d$ is an equation representation of a linear relationship between n and a_n. In this section, we are trying to identify linear relationships in various representations.

1. (Illustrative Math 8th grade) A container of fuel dispenses fuel at the rate of 5 gallons per second. If y represents the amount of fuel remaining in the container, and x represents the number of seconds that have passed since the fuel started dispensing, then x and y satisfy a linear relationship. In the coordinate plane, will the slope of the line representing that relationship have a positive, negative, or zero slope? Explain how you know.

2. (Illustrative Math 8th grade) A sandwich store charges a delivery fee to bring lunch to an office building. One office pays $33 for 4 turkey sandwiches. Another office pays $61 for 8 turkey sandwiches. How much does each turkey sandwich add to the cost of the delivery? Explain how you know.

3. (Engage NY 8th grade) A rental car company offers a rental package for a midsize car. The cost comprises a fixed $30 administrative fee for the cleaning and maintenance of the car plus a rental cost of $35 per day.

   (a) Using x for the number of days and y for the total cost in dollars, construct a function to model the relationship between the number of days and the total cost of renting a midsize car.

   (b) The same company is advertising a deal on compact car rentals. The linear function $y = 30x + 15$ can be used to model the relationship between the number of days, x, and the total cost in dollars, y, of renting a compact car. What is the fixed administrative fee? What is the rental cost per day?

4. (Connected Math Project, 8th grade - linear vs. nonlinear) Imagine a large $3 \times 3 \times 3$ cube made out of unit cubes. The large cube falls into a bucket of paint, so that the faces of the large cube are painted blue. Now suppose you broke the cube into unit cubes. How many unit cubes would be painted on:

   (a) Three faces?
   (b) Two faces?
   (c) One face?
   (d) No faces?

   What if it's a $4 \times 4 \times 4$ cube? What if it's a $15 \times 15 \times 15$ cube? How do you know, without counting?

   (You need a scratch paper to solve this question.)

5. Create a graph, a table, and an equation (if not already given) to represent the relationship between each of the following quantities. Determine whether or not the relationships between quantities are linear and justify your answer.

   (a) For a square, the length of one side and its perimeter.

   (b) For a $30-60-90$ triangle, the length of its shortest side and its hypotenuse.

   (c) For a circle, the radius and the area.

   (d) Distance traveled and a taxicab fare given the following table:

Miles	Fare
1	$5.00
2	$7.40
3	$9.80
4	$12.20

   (e) The quantities x and y, where $y = 3x$.

   (f) The quantities x and y, where $y = 4x + 2$.

   (You need a scratch paper to solve this question.)

Use your equations, tables, and graphs from the previous problem to help you answer the following questions.

1. If you have an equation describing the relationship between two quantities, how can you determine if they have a linear relationship?

2. If you have a table describing the relationship between two quantities, how can you determine if they have a linear relationship? What properties must a table have in order to represent a linear relationship?

3. If you have a graph describing the relationship between two quantities, how can you determine if they have a linear relationship? What properties must a graph have in order to represent a linear relationship?

Three bikers were training for a bike race. They start in the middle of a bike path, and each has a stopwatch which tells them when they passed various mile markers. Some of the information from their training is shown in the tables below.

Ben

Time	Mile Marker
12 min	61
36 min	67
1 hr 20m	78
1 hr 32m	81

Jing

Time	Mile Marker
15 min	16
45 min	27
1hr	30
1hr 12m	35

Amanda

Time	Mile Marker
24 min	34
36 min	39
1hr 12m	54
1hr 48m	69

1. For each biker, determine if he or she was biking at a constant speed (at least, if the speed could be constant, as far as you can tell based on the data in the tables). Provide a full explanation for how you solved this problem and why that makes sense.

2. For the bikers that were biking at a constant speed:

   (a) How long does it take them to bike one mile?

   (b) How fast were they biking?

   (c) At what mile marker will they be when their stopwatch reads 2 hours?

3. For all the bikers, can you figure out where he or she entered the trail? How did you figure this out, or why couldn't you?

Ben and Amanda are going to race. Suppose they race at the same speeds they had in training. The race course is 75 miles long. Amanda is going to start one hour after Ben.

1. Draw a graph of this situation. What variables should go on which axes? Be neat, and think ahead to find a good scale.

2. **For now**, we will say that two quantities x and y are in a linear relationship if, when we graph x vs y, we get a line. This means that x and y are in a linear relationship if there is a constant number m (the slope) and a constant number b (the y-intercept) so that $y = mx + b$.

   Why are your graphs lines above? Why would a graph of Jing's biking not be a line?

3. Use your graph to decide if Amanda will catch up with Ben by the end of the race. If so, when and where?

4. Using the graph, find out when each person will finish the race.

5. Find algebraic equations describing rules for the two bikers that tells their distance in the race at a given time. Be sure to say what your variables stand for, any special cases in which your rules don't apply, etc. Solve previous questions again using these rules.

6. What are the $x-$ and $y-$intercepts for each biker in this problem? What do they tell you in this problem situation?

7. What is the slope for each biker in this problem? What does the slope tell you in this problem situation?

Just about any relationship found in middle school algebra can be represented in the **4 ways** we have seen in this activity: we can describe the relation in terms of a **table** of values, a **graph**, an **equation**, or in terms of a **problem context** (something like "Ben bikes at the constant rate of .25 miles per minute; how long does it take him to finish a 75 mile race?). It's important for teachers to be able to recognize the strengths and weaknesses of each of these representations as well as to be able to move flexibly between them.

1. Describe a linear relationship between two quantities with a graph, an equation, and a table. What are the differences among linear relationships between two variables, linear expressions, linear equations in one, two, or more variables, equation of lines, linear functions?

2. Suppose you know that x and y are in a linear relationship. Also suppose that you know the points $(5, 12)$ and $(11, 21)$ are on the graph of this relationship.

   (a) Without finding the equation of the line, find y when x is 7. Justify your reasoning.

   (b) When $x = 3$, find y without finding the equation of the line. How about when $x = 14$?

3. Sometimes, we call this kind of reasoning "additive" reasoning. Explain how your work in the question above fits with this name.

3.4 Proportional Relationships

A proportional relationship between two quantities is a special case of a linear relationship. In this section, we will revisit proportional relationships in terms of table, graph, and equation representations. We will also investigate why a proportional relationship is a linear relationship using each representation.

[The Connected Gear Problem] - from Patterns, Quantities and Linear Functions by Amy B. Ellis, Mathematics Teaching in the Middle School, Vol. 14, No. 8, April 2009

1. You have two gears on your table. Gear A has 8 teeth, and gear B has 12 teeth. If you turn gear A a certain number of times, does gear B turn more revolutions, fewer revolutions, or the same number? How can you tell?

2. If you could replace gear A with a new gear that would make gear B turn twice as fast, how many teeth would the new gear have?

3. You want to replace gear A with a different gear to make gear B turn twice as slow instead of twice as fast. How many teeth would that different gear have?

4. Right now, gear A has 8 teeth and gear B has 12 teeth. Can you think of two different sizes for gear A and gear B so that the gear ratio would still be the same? How many possibilities can you find?

5. Gear C has 6 teeth. You hook it up to gear D and turn it a certain number of times. Gear D turned $\frac{1}{4}$ of the number of turns that gear C turned. How many teeth would gear D have to have?

6. Gear E has 15 teeth. If you turn it 12 times, gear F turns 10 times. How many teeth does gear F have to have?

7. Ricardo was working with gear A and gear B (8 teeth and 12 teeth). He turned the gears a certain number of times. He is not sure exactly how many times, because he lost count. Then he turned gear A one extra rotation. How many extra rotations did gear B turn?

8. You have two gears, one small one and one big one. The smaller gear has $\frac{2}{3}$ of the number of total teeth that the bigger gear has. How does this relate to the rotations the gear make? Justify your answer.

9. What is the relationship between the number of teeth on the gears and the number of rotations that the gear make? How many teeth does each gear have? Why?

10. The following table represents pairs of gear rotations:

# of rotations				
gear A	$7\frac{1}{2}$	27	18	12
gear B	5	18	12	8

Do all of the pairs come from the same gear combination? How can you tell? Describe the gear situation(s) that generated these pairs.

Proportional reasoning is at the heart of a lot of middle school mathematics, but sometimes, as we've already seen, students can rely so much on the algorithms that they forget why they are doing what they're doing. Of course, it's also to understand the reasons behind our methods, and to encourage our students to consider it as well. As teachers, we need to know the related definitions in order to know when our algorithms apply, and why.

Definition: We say that two quantities, represented by x and y, are **directly proportional** if $\frac{x}{y}$ is a constant.

The fraction $\frac{x}{y}$ is sometimes used to represent $x : y$ and it is also called the **rate** or **ratio** of x to y.

1. Sally would have earned \$504 for working 42 hours a week but she was absent for 4 hours. How much was she paid then?

2. Create a graph, a table, and an equation (if not already given) to represent the relationship between each of the following quantities. **Using the definition**, decide whether the following things are directly proportional.

   (a) For a square, the length of one side and its perimeter;

   (b) For a square, the length of one side and its area;

   (c) Distance traveled and the taxi fare as given in the following table;

Miles	Fare
1	\$5.00
2	\$7.40
3	\$9.80
4	\$12.20

(d) Driving speed and the amount of time it takes to drive from Madison to Chicago;

(e) The distance you jog and the time spent jogging, given that you run a 10-minute mile;

(f) Your age and my age;

(g) During a 25% off sale, the original price of an item and the sale price of the item.

3. If you have an equation describing the relationship between two quantities, how can you determine if they have a proportional relationship?

4. If you have a table describing the relationship between two quantities, how can you determine if they have a proportional relationship? What properties can you identify from the table?

5. If you have a graph describing the relationship between two quantities, how can you determine if they have a proportional relationship? What properties can you identify from the graph?

Recall that we say that two quantities represented by x and y, are **directly** proportional if $\frac{x}{y}$ is a constant. Also recall that we decided that "driving speed and the amount of time it takes to drive from Madison to Chicago" was not directly proportional. What is the relationship between speed and time here?

So what is remaining constant in this driving example?

Definition: We say that two quantities represented by x and y, are **inversely** proportional if xy (x times y) is a constant.

Let's explore some more directly and inversely proportional examples:

1. Boyle's Law says that if the temperature and the amount of air inside a balloon are fixed, the pressure P inside the balloon and the volume V of gas inside the balloon, satisfy $PV = k$ where k is some constant. If you squish the balloon, and so decrease the volume, what happens to the pressure? What sort of relationship is this?

2. Define the concentration of salt water as the amount (mass) of salt per unit of volume of water. Suppose we have a cup of salt water. We leave it sitting out, and over time, water starts evaporating, while the salt remains. Describe the relationship between concentration and volume.

3. Suppose we have a different cup. We've filled the cup with water and begin stirring in spoonfuls of salt, without changing the amount of water. What's the relationship between the amount of salt (mass) and the concentration of the salt water?

4. Again suppose we plan to go for a drive. We set out from Madison and set the cruise control. What's the relationship between the time we drive and the distance we travel?

5. Decide whether the following relationships between x and y are directly or inversely proportional.

x	y
1	12
2	6
3	4
4	3

x	y
1	2
3	6
4	8
6	12

x	y
1	-1
2	-2
3	-3
4	-4

x	y
1	-16
2	-8
4	-4
8	-2

Now, graph each of these relationships.

6. There is a "rule of thumb" for deciding whether a proportion is a direct or inverse proportion that says: *In a direct proportion, when the first quantity goes up, the second goes up too. In an inverse proportion, when the first quantity goes up, the second goes down.* What do you think? Why is this **not** always true? (Look at the graphs you just drew.) Why is it still a useful rule of thumb? How would you adjust it to make it precise?

7. It takes Jim 12 hours to paint a house.

   (a) That's a long day, so Jim plans to hire some help. Assuming the help can paint at the same rate as Jim does, how long would it take to paint a house if Jim hired one additional person to help him? Two additional people? Three additional people? How about four additional people?

   (b) What is the relationship between the total number of people painting and the time it takes to finish a house?

8. Jim and his son Charlie also mow lawns. If they mow at the same time, they can mow 3 lawns in one hour. If Jim mows for half an hour, and then Charlie takes over and mows for 2 hours, they can again mow 3 lawns. How many lawns can Jim (by himself) mow in one hour?

Recall that Ben entered the bike trail at mile marker 58 and rode at a constant speed of 3 miles every 12 minutes. Is this a proportional situation? That is, is the ratio between *Ben's mile marker* and *time* proportional (either directly or inversely)? Is it linear?

1. Are all directly proportional relationships also linear relationships? Are all linear relationships also directly proportional relationships?

2. Explain how you can tell the difference between the two in terms of tables, equations as well as in terms of graphs.

3. What about inversely proportional relationships? Can such a relationship also be linear or directly proportional? Why or why not?

4. Matt bikes at a speed of 6 mph, and Lucy bikes at a speed of 18 mph.

    (a) At 2:34pm, Matt and Lucy took off biking in opposite directions. What's the relationship between *time* and *distance between the bikers*? Write an equation to prove it.

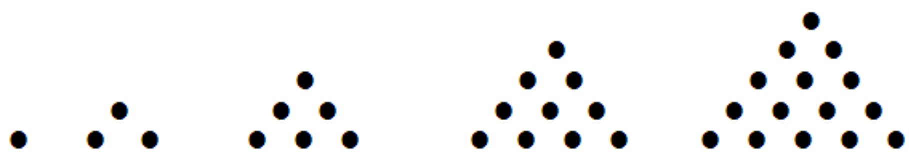
(b) Now use your equation to find the time at which they will be 64 miles apart.

(c) Now suppose Matt started biking at 2:34pm, and 20 minutes later, Lucy started biking in the opposite direction. At what time will they be 64 miles apart?

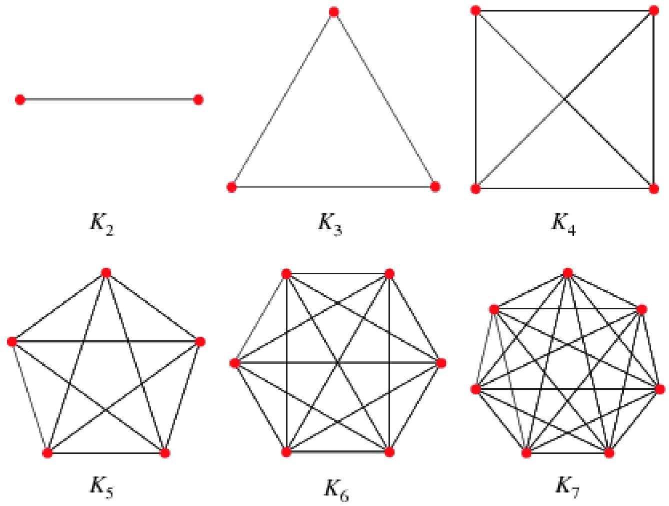
3.5 Non-linear Relationships

In the previous sections, we dealt with sequences where the differences between the terms was a constant value. In this section we extend this idea to sequences where the differences are not constant, but the second differences are constant (quadratic sequences). We also revisit geometric sequences that can be extended to exponential relationships.

1. Find a method for computing the number of dots in the nth stage. Explain and justify your method using words and equations, if possible.



2. Each diagram shows the number of line segments needed to connect a set of n points, no three of which lie in a straight line.

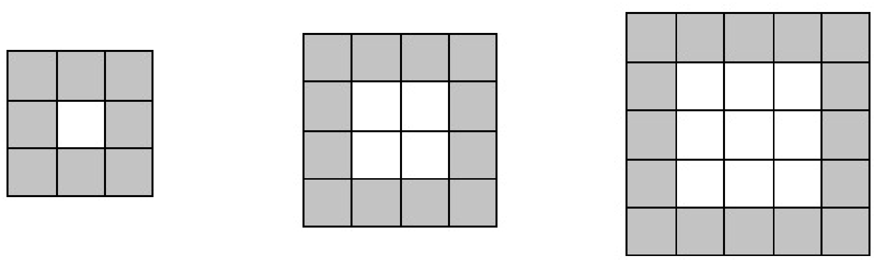


   (a) Create a sequence with at least six terms to show the relationship between the number of points and the number of line segments needed to connect every point to every other point.

   (b) Create a quadratic function $t_n = an^2 + bn + c$ to generate the sequence in part (a).

(c) Use the function in part (b) to determine the number of line segments needed to interconnect 25 points, no three of which lie in a straight line.

3. (Review of linear relationships) Consider the 1st, 2nd, and 3rd stages of the figure below.



(a) Find a recursive form for the number $b(n)$ of shaded blocks at the n^{th} stage.

(b) Find an explicit form for the number $b(n)$ of shaded blocks at the n^{th} stage.

(c) Give a sentence to explain how you know the relationship between n and $b(n)$ is linear.

(d) Give a sentence to explain how you connect the figure to your explicit form in (b).

4. At an airshow, a plane is in a power drive. The height of the plane in meters after 1, 2, 3, 4, 5, and 6 seconds is $\{71, 64, 59, 56, 55, 56\}$.

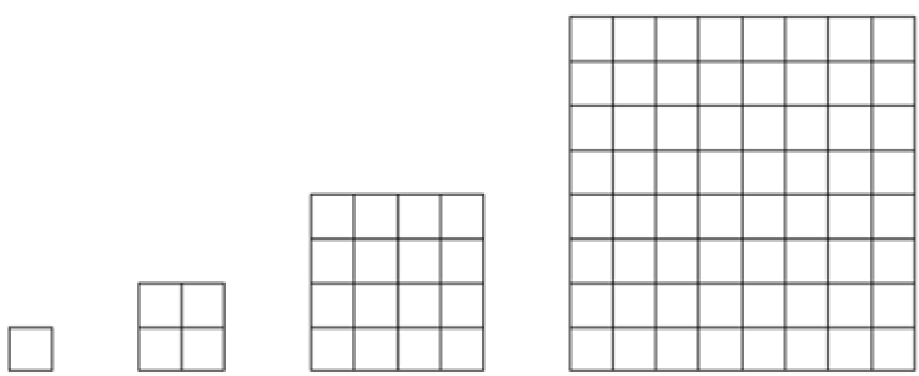
   (a) Explain and show why the sequence is a quadratic sequence.

   (b) Find an explicit form for the height $f(x)$ in meters of the plan after x seconds. You can use the general form $f(x) = ax^2 + bx + c$ of a quadratic function.

5. Find the n-th term and the first three terms of the arithmetic sequence having $a(4) = 93$ and $a(8) = 65$.

6. Let y be the amount per hour, in dollars, that Kay makes babysitting x children. Kay charges \$12 per hour for one child and an extra \$5 for each additional child. Write an equation that describe the relationship between y and x. Is the relationship between y and x linear or non-linear? Give a sentence to explain how you know the relationship is linear or non-linear.

7. Consider the 1st, 2nd, 3rd, and 4th stages of the figure below.



(a) Find a recursive form for the number $a(n)$ of blocks at the n^{th} stage.

(b) Find an explicit form for the number $a(n)$ of blocks at the n^{th} stage.

(c) Give a sentence to explain how you know the relationship between n and $b(n)$ is non-linear.

(d) Give a sentence to explain how you connect the figure to your explicit form in (b).

8. Find the n-th term $b(n)$ of the geometric sequence having $b(1) = 1$, $b(2) = -\frac{1}{2}$, and $b(3) = \frac{1}{4}$.

9. Noah put $40 on his fare card. Every time he rides public transportation, $2.50 is subtracted from the amount available on his card.

   (a) How much money, in dollars, is available on his card after he takes x rides?

   (b) Assume \$ y available on his card after he takes x rides. Is the relationship between y and x linear or non-linear? Give a sentence to explain how you know the relationship is linear or non-linear.

3.6 Ratio and Proportions Review

We say that the **ratio** between two **quantities** is $A : B$ if there is a unit so that the first quantity measures A units and the second measures B units. (In writing the ratio, one does not specify the unit.) Two ratios are **equivalent** or **equal** if one is obtained from the other by multiplying or dividing all of the measurements by the same (nonzero) number. A **proportion** is a statement that two ratios are equal.

1. If $a : b = c : d$, prove that $\frac{a}{b} = \frac{c}{d}$ using the definition of equivalent ratios.

2. The ratio of Isabel's money to Rosalind's money is $8 : 3$. If Isabel has $24, how much money do they have together? Use **a proportion**, **a bar diagram**, and/or **the definition of the ratio** to solve.

3. Express each of the following ratios as simply as possible in the form $a : b$.

   (a) A mass of 2kg to a mass of 850g.

(b) $0.4 : 1.2$

(c) A length of $\frac{3}{4}$ft to a length of $\frac{2}{3}$ft

4. Two numbers are in the ratio $3 : 5$. After subtracting 11 from each, the new ratio is $2 : 7$. What are the two numbers?

5. If $A : B = 5 : 6$ and $B : C = 4 : 5$, find $A : B : C$. The ratio between two numbers can be represented either by $a : b$ form or by $\frac{a}{b}$ form. Discuss which representation is more appropriate in this example. You may draw bar diagrams and use LCM or GCF of two numbers to figure out $A : B : C$.

6. A ribbon is cut into 3 pieces, A, B, and C, in the ratio $1:2:4$. If C is longer than B by 16cm, find the length of the ribbon.

7. The ratio of the number of boys to the number of girls in a group of students is $2:3$. If $\frac{1}{4}$ of the boys and $\frac{1}{3}$ of the girls wear glasses, find the ratio of the number of girls who wear glasses to the number of boys who wear glasses.

Percent

The expression **per cent** comes from a Latin phrase *per centum* meaning 'for each hundred' or 'out of a hundred'. Use ratios and proportions first to solve the following problems, and then use the decimal representation of percent and algebra to revisit them.

1. David saves \$600 and Justin saves \$700. Express Justin's savings as a percentage of David's.

2. A baker made 350 cookies. 100 of them were chocolate chip and the rest were sugar cookies. How many percent fewer chocolate chip cookies were there than sugar cookies?

3. Peter is 8% heavier than Henry, and James is 10% lighter than Henry. By what percentage is Peter heavier than James?

4. In the course of a year, $\frac{1}{3}$ of a company's capital increases by 4%, $\frac{1}{5}$ of it decreases by 5% and the rest increases by 10%. What is the percentage increase on the whole?

5. As a B&N member, you can get 10% discount. Sometimes B&N send you **extra** 20% discount coupons. If you apply a coupon in addition to your membership discount, can you get 30% off? What if the membership discount is 20%, and you have an **extra** 10% off coupon? Is it total 30% off? Why or why not?

Rate and speed

When a ratio is used to compare two quantities that involve different units, the result is called a **rate**. Some textbooks use different definitions. For example in CCSSM (Common Core State Standards for Mathematics), a (unit) rate is defined as a fractional form of "ratio" and it is used to describe how a quantity is changing with another quantity. $(a : b = \frac{a}{b} : 1)$. A ratio indicates what fraction on quantity is of the other, or how many times one quantity is as much as the other.

1. If a typist can type 250 words in 5 minutes, how many words can she type in 1 minute?

2. A motorist took 2 hours to travel from town X to town Y. For the first half of the journey, he traveled at an average speed of 60 km/h. His average speed for the second half was 80 km/h. Find his average speed for the whole journey.

3. A motorist took $2\frac{1}{2}$ hours to travel from town X to town Y. His average speed for the whole journey was 80 km/h. For the first $\frac{1}{4}$ of the journey, he traveled at an average speed of 60 km/h. Find his average speed for the second part of the journey.

4. A motorist took some hours to travel from town X to town Y. His average speed for the whole journey was 80 km/h. For the first $\frac{1}{4}$ of the journey, he traveled at an average speed of 60 km/h. Find his average speed for the second part of the journey. Compare this question to the previous question. What's the difference? Did you get the same answer for both or not?

5. A motorist took some hours to travel from town X to town Y. For the first $\frac{1}{2}$ of the journey, he traveled at an average speed of a km/h. His average speed for the second half of the journey was b km/h. Find his average speed for the whole journey in terms of a and b.

6. A car traveling at a uniform speed started at noon and covered the first 150 km of a journey by 3:00 pm. Find the time when it had completed the whole journey of 600 km.

Cross-multiply algorithm

You are probably familiar with the **cross-multiply algorithm**. The cross multiply algorithm is usually used when two ratios in a proportion are given in the form of $\frac{a}{b}$, not in the form of $a : b$. Teachers and students usually don't know why and how this algorithm work. The following problems help us understand the meaning of the algorithm.

1. If $a : b = c : d$, prove that $ad = bc$. (Hint: We proved $a : b = c : d$ implies $\dfrac{a}{b} = \dfrac{c}{d}$.)

2. A recipe for cookies calls for 1 cup of sugar for 24 cookies.

   (a) How much sugar should you use for 36 cookies?

   (b) What about for 32 cookies?

3. How did you know how to answer the questions above? What assumptions are you implicitly making?

4. Think of at least two different ways to approach solving these problems. Discuss your methods. Do you think one of the methods would be better for younger children?

5. Consider the following student's (correct!) work below and explain what each step and each part of their equations mean. Be sure to explain why the first equation the student wrote is correct and makes sense. Also, think carefully about step 3.

$$
\begin{aligned}
\frac{1 \text{ cups of sugar}}{24 \text{ cookies}} &= \frac{x \text{ cups of sugar}}{36 \text{ cookies}} && (1)\\
36 \text{ cookies} \times \frac{1 \text{ cups of sugar}}{24 \text{ cookies}} &= x \text{ cups of sugar} && (2)\\
36 \text{ cookies} \times \frac{1 \text{ cups of sugar}}{24 \text{ cookies}} &= x \text{ cups of sugar} && (3)\\
\frac{36 \text{ cups of sugar}}{24} &= x \text{ cups of sugar} && (4)\\
1.5 \text{ cups of sugar} &= x \text{ cups of sugar} && (5)
\end{aligned}
$$

6. Consider the following student's work and explain each step of their reasoning and why each equal sign works.

$$\frac{1 \text{ cups of sugar}}{24 \text{ cookies}} = \frac{x \text{ cups of sugar}}{32 \text{ cookies}} \tag{1}$$
$$\frac{1}{24} = \frac{x}{32} \tag{2}$$
$$24x = 32 \tag{3}$$
$$x = \frac{32}{24} \tag{4}$$
$$x = 1\frac{1}{3} \tag{5}$$

7. A certain shade of green paint was created by mixing 2 cans of blue paint with 7 cans of yellow paint.

   (a) Which shade of green is "bluer": the one above, or one made from 3 cans of blue paint and 12 cans of yellow paint?

   (b) If we have 9 cans of blue paint and want to recreate the same shade of green, how many cans of yellow paint will we need? Answer this question using the cross-multiply algorithm.

(c) If we instead start with 4 cans of blue paint, how many cans of yellow paint will we need in order to get the same green? Can you use a picture to explain your solution?

(d) What are the limitations of this pictorial method? What are the advantages? Think about if you could use the method to answer the problem if we started with 20 cans of blue paint, or $\frac{1}{3}$ of a can, or 2.15 cans.

(e) Think about this technique graphically. Draw a graph with "cans of blue paint" on the horizontal axis and "cans of yellow" paint on the vertical axis. Plot your answers for parts (b) and (c) on this graph. What is the interpretation of slope in this context? Estimate the cans of yellow paint needed if we start with $\frac{1}{3}$ of a can of blue paint. What are the limitations and advantages of this method?

(f) Think about this technique algebraically. Generalize your technique so that you can find the number of cans of yellow paint required for any given number of cans of blue paint. What if we started with $\frac{1}{3}$ of a can of blue paint? 20 cans of blue paint? 2.15 cans of blue paint? What are the limitations and advantages of this method?

(g) Can you connect this generalized technique to the cross multiply algorithm? Why do they give the same answer?