Math 491 - Linear Algebra II, Fall 2015

Practice Final

May 3, 2016

Remarks

• Answer all the questions below. The best three (a), (b), and (c) will be counted towards your score.

• A definition is just a definition - there is no need to justify it. Just write it down.

• Unless it’s a definition, answers should be written in the following format:
  – Write the main points that will appear in your proof of computation. Main points:...
  – Write the actual explanation or proof or computation. Proof:... or Computation:...

1. Jordan Canonical Form

(a) (8) Define the following types of matrices:
   (i) A Jordan block $J_r(\lambda) \in M_r(\mathbb{C})$ associated to eigenvalue $\lambda$;
   (ii) A Jordan array $J(\lambda) \in M_k(\mathbb{C})$ associated to eigenvalue $\lambda$ and the notion of index of an array;
   (iii) A Jordan matrix $J \in M_n(\mathbb{C})$.

State precisely the Jordan canonical form theorem for an operator $T : V \to V$ where $V$ is an $n$-dimensional vector space over $\mathbb{C}$.

(b) (15) Assume that $V$ is a vector space over $\mathbb{C}$ of dimension 4. Let $T : V \to V$ be a linear transformation.

   (i) Recall that there exists a Jordan basis $B$ of $V$ such that $[T]_B = J$ where $J \in M_4(\mathbb{C})$ is a Jordan matrix. Show that, up to permutation of the Jordan blocks, $J$ is the unique Jordan matrix for which a Jordan basis for $T$ exists.

   (ii) Compute a Jordan basis for the transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T_A(v) = Av$, where

   $A = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{pmatrix}$.
(c) (10) Let $T, S : V \rightarrow V$ be two linear transformations on a finite dimensional vector space $V$ over $\mathbb{F}$. Suppose that $m_T(x) = m_S(x)$, and
\[ p_T(x) = p_S(x) = (x - \lambda_1)^{d_1} \cdots (x - \lambda_k)^{d_k}. \]
Suppose the $\lambda_i \in \mathbb{F}$ are distinct, and for all $1 \leq i \leq k$, $1 \leq d_i \leq 3$. Show that $T$ and $S$ are similar.

2. Inner Product Spaces

(a) (8) Define the notion of an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(b) (15) Define when two inner product spaces are isometric. Show that if $(V, \langle \cdot, \cdot \rangle)$ is an $n$-dimensional inner product space then it is isometric to $(\mathbb{F}^n, \langle \cdot, \cdot \rangle_{\text{st}})$.

(c) (10) Let $V = \mathbb{R}_{\leq 3}[x]$ be the vector space of real polynomials with degree at most 3. Given two polynomials, $f, g \in V$, recall that the pairing
\[ \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx \]
defines an inner product on $V$. Let $W = \text{Span}\{x^2, x + 1, x^3\}$. Use the Gram-Schmidt process to find an orthonormal basis for $W$.

3. Adjoint Operator

(a) (8) Let $T$ be a linear transformation on a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. Define the adjoint operator $T^* : V \rightarrow V$.

(b) (15) Let $T$ be a linear transformation on a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. Show that $T^*$ is unique and prove its existence.

(c) (10) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $A \in M_n(\mathbb{F})$ define the matrix $A^*$ and write a formula for it in terms of $A$. Let $(V, \langle \cdot, \cdot \rangle)$ be an $n$-dimensional inner product space over $\mathbb{F}$. Show that if $T : V \rightarrow V$ is a linear transformation and $\mathcal{B} \subset V$ is an orthonormal basis for $V$ then
\[ [T^*]_{\mathcal{B}} = [T]^*_{\mathcal{B}}. \]

4. Spectral Theorem

Throughout this problem, assume that $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over $\mathbb{C}$.

(a) (8) Define the notion of a normal operator $T : V \rightarrow V$.

(b) (15) State and then prove the spectral theorem for an operator $T$ on $V$. 

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(c) (10) Let $V = \mathbb{C}(F_5)$ be the vector space of complex valued functions on $F_5$. Given two vectors $f, g \in V$, the pairing

$$\langle f, g \rangle = \sum_{x \in F_5} f(x)\overline{g(x)},$$

defines an inner product on $V$. Let $\mathcal{F} : V \to V$ be the discrete Fourier transform, i.e. the linear transformation defined by

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{5}} \sum_{x \in F_5} e^{\frac{2\pi i}{5} \cdot xy} f(x).$$

Show that $\mathcal{F}$ is unitary and diagonalizable.