Math 491 - Linear Algebra II, Fall 2016

Homework 6 - Characteristic and Minimal Polynomials

Quiz on 3/15/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

Theoretical Exercises

1. Minimal Polynomials and Diagonalizability. Let $T : V \rightarrow V$ be a linear transformation on a finite dimensional vector space $V$ over $\mathbb{F}$. Recall that the minimal polynomial of $T$, denoted $m_T(x)$, is the unique monic polynomial of minimal degree that annihilates $T$, i.e. $m_T(T) = 0$. Moreover, $m_T(x)$ divides the characteristic polynomial, $p_T(x)$, which also annihilates $T$. The minimal polynomial gives the following characterization of a transformation being diagonalizable.

**Theorem** $T : V \rightarrow V$ is diagonalizable if and only if $m_T(x) = (x - \lambda_1) \cdots (x - \lambda_k)$ for distinct $\lambda_i \in \mathbb{F}$.

For each of the following matrices $A$, compute $p_A(x), m_A(x)$, and use the above theorem to decide whether $A$ is diagonalizable:

\[
\begin{pmatrix}
3 & 0 \\
1 & 3
\end{pmatrix}, \begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}, \begin{pmatrix}
3 & 0 \\
1 & 2
\end{pmatrix}, \begin{pmatrix}
1 & 2 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]


(a) Assume $A \in M_3(\mathbb{C})$ has minimal polynomial $m_A(x) = x^6 - 4x^4 + 3x^2 + 1$. Find the minimal polynomial of the matrix $A^2$.

(b) Assume $A \in M_4(\mathbb{C})$ has characteristic polynomial $p_A(x) = x^4 + 3$. Find the characteristic polynomial of the matrix $A^2$.

(c) Assume $A \in M_2(\mathbb{C})$ has minimal polynomial $m_A(x) = x^2 + x + 1$. Find the minimal polynomial of the matrix $A^2$. 

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3. **The Centralizer of an Operator.** Let \( T : V \to V \) be a linear transformation of an \( n \)-dimensional vector space over \( \mathbb{F} \). Assume \( T \) has \( n \) distinct eigenvalues.

(a) Let \( S : V \to V \) be a linear transformation such that \( ST = TS \). Show that \( S \) is diagonalizable.

(b) Recall that \( \mathcal{L}(V) \) is the algebra (i.e., it has a product in addition to a vector space structure) of linear transformations from \( V \) to itself. Define the centralizer of \( T \) in \( \mathcal{L}(V) \) by

\[
Z(T) = \{ S \in \mathcal{L}(V) \mid ST = TS \}.
\]

Show that \( Z(T) \) is a commutative subalgebra of \( \mathcal{L}(V) \), i.e. show that

(i) \( Z(T) \) is a subspace of \( \mathcal{L}(V) \), i.e., it is closed under addition, scalar multiplication, and \( 0 \in \mathcal{L}(V) \in Z(T) \).

(ii) \( Z(T) \) is a subalgebra, i.e., it is closed under multiplication and \( Id_V \in Z(T) \).

(iii) \( Z(T) \) is commutative, i.e., for \( S_1, S_2 \in Z(T) \) we have \( S_1S_2 = S_2S_1 \).

Finally, show that \( \dim Z(T) = n \).

(c) Assume now that \( T \) is diagonalizable (although it may not have \( n \) distinct eigenvalues). What can you say about \( \dim Z(T) \)?

4. **Working in \( \mathbb{C} \) to get information in \( \mathbb{R} \).** This exercise outlines two different proofs of the same result.

(a) Let \( A \in M_2(\mathbb{R}) \) and suppose \( p_A(x) = x^2 + 1 \). Show there exists an invertible matrix \( P \in M_2(\mathbb{R}) \) such that

\[
C^{-1}AC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Hint: First, show that there is an invertible matrix \( C \in M_2(\mathbb{C}) \) such that

\[
AC = C \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Now, recall the fact that a linear system defined over \( \mathbb{R} \) that has a solution in \( \mathbb{C} \), also has a solution in \( \mathbb{R} \).

(b) Let \( T : V \to V \) be a linear transformation of a 2-dimensional vector space \( V \) over \( \mathbb{R} \). Assume the characteristic polynomial of \( T \) is \( p_T(x) = x^2 + 1 \). Show that there exists a basis \( B \) of \( V \) such that

\[
[T]_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Hint: Fix a nonzero vector \( v \in V \) and consider the vectors \( v, Tv \).
Computational Exercises

5. **Verifying Cayley-Hamilton.** As a warm-up, use Matlab to verify the Cayley-Hamilton Theorem for the matrices appearing in problem 1. That is, input each of the matrices into Matlab as $A$, and the verify that $p_A(A) = 0$.

6. **The Power Method.** In many physical applications or dynamic systems, the largest eigenvalue associated to a matrix represents the dominant and most interesting mode of behavior. The Power Method is a naive algorithm that attempts to compute the largest magnitude eigenvalue. Specifically, the algorithm runs like this.

   1. **Input:** a matrix $A \in M_n(\mathbb{R})$ and a fixed number $N$ of steps desired
   2. **Initialize:** choose a random unit vector $x^0 \in \mathbb{R}^n$ and set $r_0 = 0$
   3. **Iterate:** for $k = 0, 1, 2, \ldots, N$
      - $x^{k+1} := \frac{Ax^k}{||Ax^k||}$
      - $r_{k+1} := \frac{(x^k)^\top Ax^k}{||x^k||^2}$
   4. **Output:** a unit vector $x^N$ and number $r_N$

   (a) Write a Matlab m-file that implements the above algorithm. Discuss whether your algorithm is working by considering the output when it runs with input $N = 100$ and

   $A = \begin{pmatrix} 0 & -1 & 1 \\ 7 & 5.5 & -7 \\ 5 & 2.5 & -4 \end{pmatrix}$.

   (b) Assume now that the input matrix $A$ is diagonalizable and that it has a unique eigenvalue of largest magnitude. Moreover, assume that the random initial unit vector has a nonzero component when projected onto this eigenspace. Show that the output of the Power Method converges (up to a sign) to an eigenvector corresponding to this largest eigenvalue.

**Remark**
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

**Good luck!**