1. **Degree of a polynomial.** Let $\mathbb{F}$ be a field and $\mathbb{F}[X]$ the ring of polynomials with coefficients in $\mathbb{F}$.

   (a) Show that $\dim \mathbb{F}[X] = \infty$.

   (b) Now let $R$ be a ring and $R[X]$ the ring of polynomials with coefficients in $R$. Recall that the degree $\deg(f)$ of a polynomial $f \in R[X]$ is defined to be $\deg(f) = d$ if $f = a_d X^d + \ldots + a_1 X + a_0$ and $a_d \neq 0$, and $\deg(f) = -\infty$ if $f = 0$. Show that for every $f, g \in R[X]$ we have

   (i) $\deg(fg) \leq \deg(f) + \deg(g)$,

   (ii) $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

   Moreover, show that if $R$ is an integral domain than the inequality in (i) is actually an equality.

2. **Kernel of a homomorphism.** Let $\varphi : R \to S$ be homomorphism of rings. Define the kernel of $\varphi$, denoted $\ker(\varphi)$, by $\ker(\varphi) = \{a \in R \text{ such that } \varphi(a) = 0\}$. Show that $\varphi$ is one-to-one if and only if $\ker(\varphi) = \{0\}$.

3. **Uniqueness of inverses.** Show that if $R$ is a ring with unit and $a \in R$ is invertible, i.e., there exists $b \in R$ such that $ab = ba = 1_R$, then it has a unique inverse.

4. **Subrings from ring homomorphisms.** Let $\varphi : R \to S$ be a homomorphism of rings. We define the image of $\varphi$, denoted $\text{Im}(\varphi)$, by $\text{Im}(\varphi) = \{\varphi(r) \mid r \in R\}$. Show that $\text{Im}(\varphi)$ is a subring of $S$ and that $\ker(\varphi)$ is a subring of $R$.

5. **Kernel is an ideal.** Let $R$ be a ring. A subset $I \subset R$ is called an ideal of $R$ if $I$ satisfies

   (i) $0_R \in I$
(ii) for all $a, b \in I$, we have $a + b \in I$

(iii) for all $a \in I$, we have $-a \in I$

(iv) for all $a \in I$ and $r \in R$ we have $r \cdot a \in I$

(v) for all $a \in I$ and $r \in R$ we have $a \cdot r \in I$

Now, let $\varphi : R \to S$ be a homomorphism of rings. Show that $\ker(\varphi) \subset R$ is an ideal of $R$.

6. **Inverse of a Homomorphism.** Let $\varphi : R \to S$ be a homomorphism of rings. We say that $\varphi$ is **invertible** if there exists a ring homomorphism $\sigma : S \to R$ such that $\sigma \circ \varphi = \text{id}_R$ and $\varphi \circ \sigma = \text{id}_S$. Show that the following are equivalent:

   (i) $\varphi$ is invertible
   (ii) $\varphi$ is one-to-one and onto

   **Hint for (ii) implies (i):** Note that $\varphi^{-1} : S \to R$ exists. Show that $\varphi^{-1}$ is a homomorphism of rings.

7. **Division Algorithm.** Let $F$ be a field. Recall from class, if $f, g \in F[X]$ then there exists unique polynomials $q, r \in F[X]$ with $0 \leq \deg(r) < \deg(g)$ such that

   $$f = qg + r.$$

Find $q, r$ in the following cases.

   (i) Let $F = \mathbb{F}_7$, the field with 7 elements, and take $g = X^3 + X + 1$, $f = X^5 + 2X^4 - 3X^3 + X^2 - 1$.
   (ii) Let $F = \mathbb{F}_2$, the field with 2 elements, and take $g = X + 1$, $f = X^3 + X$.

8. **Ring homomorphisms and Inverses.** Let $R, S$ be rings with unit. Let $\varphi : R \to S$ be a homomorphism of rings with unit, that is $\varphi(1_R) = 1_S$. Recall that $R^\times = \{a \in R \mid a \text{ is invertible}\}$. Show that for every $a \in R^\times$, we have $\varphi(a) \in S^\times$. That is, $\varphi$ maps invertible elements to invertible elements.

9. **Ideals in Polynomial Rings.** Let $F$ be a field and $R = F[X]$. Decide whether or not the following subsets of $R$ are ideals.

   (a) $\{f \in R \mid \deg(f) < m\} \cup \{0\}$.
   (b) $\{f \in R \mid \deg(f) > m\} \cup \{0\}$.
   (c) $\{f = \sum a_iX^i \mid a_0 = 0\}$.
   (d) $\{f \in R \mid f(1) = 0\}$.
   (e) $\{f \in R \mid f(0) = 0\}$.
   (f) $\{f \in R \mid f(0) = f(1)\}$.
(g) \{ f \in R \mid f(0) = f(1) = 0 \}.

**Remark**
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

*Good luck!*