Math 491 - Linear Algebra II, Spring 2016

Homework 1 - Warm-up

Due: 2/2/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. The Change of Basis Matrix. Let \( V \) be an \( n \)-dimensional vector space over a field \( \mathbb{F} \).
   
   (a) Recall the definition of a basis of \( V \) and the coordinate map induced by a basis.
   
   (b) Suppose \( B \) and \( C \) are two bases for \( V \). Show that there exists a unique matrix \( M_{C,B} \in M_n(\mathbb{F}) \) such that, for every \( v \in V \),
   
   \[
   [v]_C = M_{C,B}[v]_B.
   \]
   
   Let \( B = \{v_1, \ldots, v_n\} \). Show that the matrix \( M_{C,B} \) has the formula
   
   \[
   M_{C,B} = \begin{pmatrix} [v_1]_C & \cdots & [v_n]_C \end{pmatrix}.
   \]
   
   (c) Let \( V = \mathbb{R}_{\leq 2}[x] \) be the vector space of polynomials of degree at most two with coefficients in \( \mathbb{R} \). Let \( B = \{1, x, x^2\} \) and \( C = \{1 - x, 1 + x, 1 + x^2\} \) be two bases of \( V \). Compute \( M_{C,B} \) and \( M_{B,C} \).

2. The Matrix of a Linear Transformation. Let \( V \) be an \( n \)-dimensional vector space over a field \( \mathbb{F} \).
   
   (a) Recall the definition of a linear transformation \( T : V \to V \).
   
   (b) Let \( T : V \to V \) be a linear transformation. Show that there exists a unique matrix \( [T]_B \in M_n(\mathbb{F}) \) such that for all \( v \in V \)
   
   \[
   [T(v)]_B = [T]_B[v]_B.
   \]
Moreover, show that the matrix \([T]_B\) has the formula
\[
[T]_B = \begin{pmatrix}
[T(v_1)]_B & \cdots & [T(v_n)]_B
\end{pmatrix}.
\]

(c) Let \(B\) and \(C\) be two bases of \(V\). Show that
\[
[T]_C = M_{C,B} [T]_B M_{B,C}.
\]

(d) Let \(V = \mathbb{R}_{\leq 2}[x]\). Let \(T : V \to V\) be given by the equation
\[
T(p(x)) = (xp(x))'.
\]
Compute \([T]_B\) and \([T]_C\) where \(B = \{1, x, x^2\}\) and \(C = \{1 - x, 1 + x, 1 + x^2\}\).

3. **Direct sum.** We say that a vector space \(V\) is a direct sum of the subspaces \(V_1, \ldots, V_k \subset V\), and denote \(V = V_1 \oplus \cdots \oplus V_k\), if every \(v \in V\) can be written uniquely as \(v = v_1 + \cdots + v_k\), where \(v_i \in V_i\) for every \(i = 1, \ldots, k\).

(a) Suppose \(V\) is a finite dimensional vector space and \(V_1, \ldots, V_k \subset V\) are subspaces. Show that the following are equivalent:
1. \(V = V_1 \oplus \cdots \oplus V_k\).
2. For every collection of bases \(B_1\) for \(V_1\), ..., \(B_k\) for \(V_k\), their union \(B = B_1 \cup \cdots \cup B_k\) is a basis for \(V\).

(b) (i) Let \(V = F(\mathbb{R})\) be the vector space of functions on \(\mathbb{R}\). Show that \(V\) is the direct sum of the subspaces generated by odd and even functions.

(ii) Let \(V = M_n(F)\) be the vector space of \(n \times n\) matrices over \(F\). Show that \(V\) is the direct sum of the subspaces generated by symmetric and anti-symmetric matrices.

4. **Eigenvalues and eigenvectors.** Let \(T : V \to V\) be a linear transformation.

(a) Recall the definition of eigenvectors and eigenvalues.

(b) Show that if \(v_1, \ldots, v_k \in V\) are eigenvectors of \(T\) associated with different eigenvalues \(\lambda_1, \ldots, \lambda_k\), then they are linearly independent.

5. **Diagonalization** Let \(V\) be an \(n\)-dimensional vector space over \(F\). Let \(T : V \to V\) be a linear transformation. Show that the following are equivalent:

1. There exists a basis \(B = \{v_1, \ldots, v_n\}\) for \(V\) such that
\[
[T]_B = \begin{pmatrix}
\lambda_1 \\
& \ddots \\
& & \lambda_n
\end{pmatrix}.
\]
2. There exists \( \mu_1, \ldots, \mu_k \in \mathbb{F} \) and subspaces \( V_1, \ldots, V_k \) of \( V \), such that \( V = V_1 \oplus \cdots \oplus V_k \), and for each \( i = 1, \ldots, k \) the action of \( T \) on \( V_i \) is given by multiplication by \( \mu_i \).

Remark
The lecturer and grader will be happy to help you with the homework. Please attend their office hours.