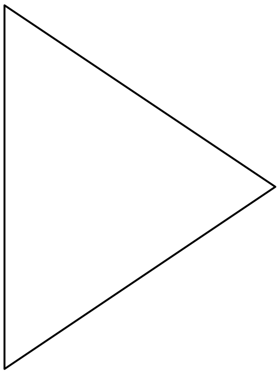
Wednesday, February 9, 2011 Lecture 6

$\mathbf{R}$—real number

R— rotations of plane

This is a triangle. How can we find the vertices of this triangle?

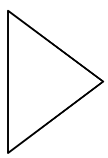
First we can draw a picture to figure out the problem.



Because, $Z^3 = 1$

So, we can calculate the position of each point:

{ [cos(2π/3)k], [sin(2π/3)k] } for k=0,1,2

Now, suppose we have this  triangle in $\mathbf{R^2}$—"space".

We can see that there is a group of rotation R ={ $r_\theta = \begin{pmatrix} con\Theta & -sin\Theta \\ sin\Theta & cos\Theta \end{pmatrix}$, $\Theta \in \mathbf{R}$}

Group operation: ∘ $--$composition of matrices

Identity element: I=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let's consider map $\cdot :$ $R \times \mathbf{R^2} \to \mathbf{R^2}$

Which is given by (r_Θ , $\begin{pmatrix} x \\ y \end{pmatrix}$)) $\longmapsto$ $r_\Theta \begin{pmatrix} x \\ y \end{pmatrix}$

We call this is an action of R on $\mathbf{R^2}$ and we can denote $\begin{pmatrix} x \\ y \end{pmatrix} = v$ here.

It satisfies:

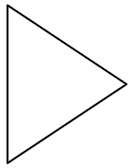
1. Associativity:

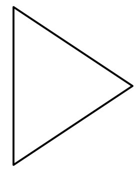
$r_{\theta 1} \cdot (r_{\theta 2} \cdot v) = (r_{\theta 1} \circ r_{\theta 2}) \cdot v$

2. Identity:

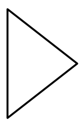
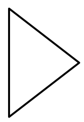
$I \cdot v = v \quad \forall\, v \in \mathbf{R}^2$

3. we have the object in $\mathbf{R}^2$

 $\subset \mathbf{R}^2$

We define the **rotational symmetries** of the triangle: rotation symmetries of 

as the set

$C_3 = \{ r \in R \mid r($  $) =$  $\}$

We see that $C_3 = \{r_0, r_{120}, r_{240}\} \subset R$

We checked in a previous lecture that $(C_3, \circ, I)$ is a group.

We have **abstract definitions**:

1. Let X represent a set of elements

2. Let $(G, \circ, 1_G)$ represents a group

Definition: we say that the group $(G, \circ, 1_G)$ acts on the set X if we have a map :

$$\cdot : \quad G \times X \to X$$

$$(g, x) \mapsto g \cdot x$$

Such that :

1. $\forall\, g, h \in G; \forall\, x \in X.$

   $g \cdot (h \cdot x) = (g \circ h) \cdot x$

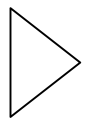
2. $1_G \cdot x = x$ for $\forall\, x \in X.$

We have two examples.

1. Consider the set X=**$\mathbf{R}^2$** and the group G= R.

We can think of it as operating by multiplication of a matrix on a vector.

So we can conclude that we have a map · : G × X→ X, and(G,∘ , 1_G) acts on the set X via this map.

2. Consider the set Y=  ⊂ **$\mathbf{R}^2$** and

C_3 = $Stab_G$(Y)

We know that C_3 = { r_0, r_{120}, r_{240} }

given by (r_θ , y) ⟼ r_θ(y) .

In a more **<u>general situation</u>**

1. Y ⊂ X

Y is the subset of X

2. G acts on X

<u>Definition</u>: The G- Symmetries of Y is the subset of G ⊃ { g∈ G| g (y) =Y} named $Stab_G$(Y) " stabilizer subgroup", where g (Y)= {g ·y | y∈ Y }.

Claim: The ∘ operation of G induces a natural group structure on $Stab_G$(Y).

i.e. ,

∘ is an operation on $Stab_G$(Y) and the triple ($Stab_G$(Y), ∘, 1_G) is a group.

In order to help understand more about the definition of "subgroup", there is another simpler example.

Define:

Let (G, $*_G$, 1) be a group.

A subset H ⊂G is called **<u>subgroup</u>** of G, if it is a group with respect to the operation $*_G$

we can conclude that (H, $*_G$, 1) is a group and H is a subgroup of G, notated by H < G.

Claim: $(Stab_G(Y), *_G, 1_G)$ is a subgroup of G, denoted by $Stab_G(Y) < G$.

Proof:

1. Closure.

So let's take $g, h \in Stab_G(Y)$ to show that $g *_G h \in Stab_G(Y)$

$(g *_G h) (Y) = g *_G (h *_G Y) = g(Y) = Y$

The first equality is because $h \in Stab_G(Y)$

Therefore, we have $(h *_G Y) = Y$.

The second equality is because $g \in Stab_G(Y)$

Therefore, we have $g (Y) = Y$.

So we cam conclude that $g *_G h \in Stab_G(Y)$

2. Identity

$1_G \in Stab_G(Y)$

$1_G *_G Y = Y$

3. Associativity

Suppose $g_1, g_2, g_3 \in Stab_G(Y) \subset G$

$g_1 *_G (g_2 *_G g_3)=(g_1 *_G g_2) *_G g_3$

$\because g_1, g_2, g_3$ are in G, $\therefore$ the associativity also holds.

4. Inverse:

$g^{-1} (Y) = (g^{-1})(g(Y))$ $\leftarrow g \in Stab_G(Y)$

$=(g^{-1})(g(Y)) = (g^{-1} *_G g) (Y)$ $\leftarrow$ by associativity

$= (g^{-1} *_G g) (Y) =1_G(Y)$ $\leftarrow$ by inverse

$= 1_G(Y) = Y$ $\leftarrow$ by identity

overall, we obtained $g^{-1} (Y)= Y$

To conclude, $Stab_G(Y) < G$ ∎