Monday, February 7, 2011 Lecture 5

Continue with the homework 1 problem 3 (c).

Permutations:

$X_3=\{a_1, a_2, a_3\}$

$X=\{a_1, a_2, \dots \dots, a_n\}$

$Aut(x)=S_n=\{\sigma: X \to X; \sigma \text{ is a bijection }\}$

$\circ$ operation of standard composition of functions

$id(x)=x, \forall x \in X.$

Claim:

$\#S_n= n!$ the number of elements of S_n is n!.

$\#S_3=3! = 6$ the number of elements of S_3 is 6.

Conventional:

$$\begin{array}{ll} \text{Domain} & \\ \sigma & \\ \text{Range} & \end{array} \begin{pmatrix} a_1 & a_2 & \dots \dots & a_n \\ & & & \\ \sigma(a_1) & \sigma(a_2) & \dots \dots & \sigma(a_n) \end{pmatrix}$$

for example with n=3:

$$\begin{array}{ll} \text{Domain} & \\ \sigma & \\ \text{Range} & \end{array} \begin{pmatrix} a_1 & a_2 & a_3 \\ & & \\ \sigma(a_1) & \sigma(a_2) & \sigma(a_3) \end{pmatrix}$$

Proof of (c):

We know for each element in the domain we have to send it to one element in the range. Since the function is one to one and onto, for instance by firstly looking at a_1 element, we'll have

$\underline{\mathbf{a_1}}$ corresponding to **n** options to send it to the range;

$\underline{\mathbf{a_2}}$ corresponding to **n-1** options to send it to the range;

… …

$\underline{\mathbf{a_n}}$ eventually only left with **1** option to send it to the range.

Overall, we compute the total ways of sending each element in the domain to the range, we obtain n*(n-1)*(n-2)*…*3*2*1=n! (by definition)

Similarly, for n=3, S_3=3*2*1=6.

3(d)

Write down all the elements of S_3.

$$\begin{pmatrix} a1 & a2 & a3 \\ a1 & a2 & a3 \end{pmatrix}, \begin{pmatrix} a1 & a2 & a3 \\ a1 & a3 & a2 \end{pmatrix}, \begin{pmatrix} a1 & a2 & a3 \\ a2 & a1 & a3 \end{pmatrix}, \begin{pmatrix} a1 & a2 & a3 \\ a2 & a3 & a1 \end{pmatrix}, \begin{pmatrix} a1 & a2 & a3 \\ a3 & a1 & a2 \end{pmatrix}, \begin{pmatrix} a1 & a2 & a3 \\ a3 & a2 & a1 \end{pmatrix} \blacksquare$$

3(e)

Definition: A group (G, ∘, 1_G) is called **commutative** ,

if ∀ g, h ∈ G, g ∘h= h ∘ g .

If ∃ g, h ∈ G, g ∘h ≠ h ∘ g, then we say that G is a **non-commutative** group.

Example: The group (Z, +, 0) of integers with addition operation and zero as identity, is **commutative**.

m +n = n+ m   ∀ m, n ∈Z.

Claim: S_3 (or even to be more in general ∀n ≥3) is a **non-commutative** group.

Compute:

$\sigma_{a,b} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$ $\sigma_{a,c} = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$

Let's calculate:

$\sigma_{a,b} \circ \sigma_{a,c}$

$= \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \circ \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$

However,

$\sigma_{a,c} \circ \sigma_{a,b}$

$= \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$

Therefore, we have $\sigma_{a,b} \circ \sigma_{a,c} \neq \sigma_{a,c} \circ \sigma_{a,b}$

∎

How would you show that ∀ n ≥ 3, S_n is **non-commutative**?

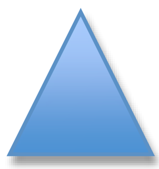
Simply, we can use a counter example to show that ∀ n ≥ 3, S_n is non-commutative.

$$\begin{pmatrix} a1 & a2 & a3 & a4 & \ldots\ldots & an \\ \sigma(a1) & \sigma(a2) & \sigma(a3) & a4 & \ldots\ldots & an \end{pmatrix}$$

where $\sigma_{a1,a2} \circ \sigma_{a1,a3} \neq \sigma_{a1,a3} \circ \sigma_{a1,a2}$.

Rotational Symmetries

I)



$\subset \mathbf{R}^2 = \{ (x, y) \mid x, y \in \mathbf{R}\}$

II) $(R, \circ, 1_R)$ ←this is a group

$R = \{ r_\theta = \begin{pmatrix} con\Theta & -sin\Theta \\ sin\Theta & cos\Theta \end{pmatrix}, \Theta \in \mathbf{R}\}$

$\circ$ ← composition of operators (matrix multiplication)

and $r_{\theta1} \circ r_{\theta2} = r_{\theta1+\theta2}$

$r_\theta \circ r_{-\theta} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

III) Note that we have a map:

$\cdot : R \times \mathbf{R}^2 \to \mathbf{R}^2$ action of R on $\mathbf{R}^2$

$(r_\theta, v) \longmapsto r_\theta \cdot v = r_\theta(v)$ the rotation of v by θ

Properties of ·

1. $r_{\theta2} \cdot (r_{\theta1} \cdot v) = (r_{\theta2} \circ r_{\theta1}) \cdot v$ it holds true for multiplications of any 3 matrices.

$\forall\, \theta_1, \theta_2, v \in \mathbf{R}^2$.

2. $1_R \cdot v = v$, $\forall\, v \in \mathbf{R}^2$

IV) Symmetries

$C_3 = \{ r_\theta \in R \mid r_\theta(\triangle) = \triangle\}$

It is a group.