Math 541
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February 16, 2011

Definition: We say that H, a subset of G, is a **<u>subgroup</u>** if H is a group with a special element 1_H with respect to the operation $\circ$ of G.

Notation: $H<G$.

Theorem:

Claim: $H<G$ iff

(I) H is none empty.
(II) For every $h_{1,} h_2 \in H$, we have $h_1 \circ h_2 \in H$ for operation.
(III) For every $h \in H$, its inverse $h^{-1} \in H$.

Proof: $\Longrightarrow$

Assume that $H<G$
(I) H is none empty because $1_H \in H$
(II) $h_1 \circ h_2$ is in H because of closure.
(III) Consider this $h^{-1} \in G$ s.t.
$h \circ h^{-1} = 1_G$

Lemma: $1_G = 1_H$
Proof:
$1_H \circ 1_H = 1_H$
There exists $1_H^{-1} \in G$ for $1_H \in G$,
$1_H^{-1} \circ (1_H \circ 1_H) = 1_H^{-1} \circ 1_H$
$1_H = 1_G$

Therefore, $h \circ h^{-1} = 1_H$
Let h' be the inverse of h in H
$h' \circ (h \circ h^{-1}) = h' \circ 1_H$
$h^{-1} = h' \in H$

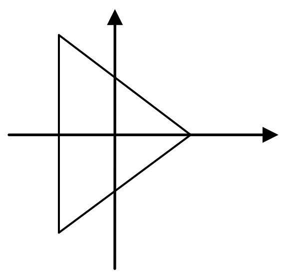
Proof: $\Longleftarrow$

Assume H is none empty and it's in G satisfies (I) (II) (III).
We want to show that H is a subgroup of G.
Closure: Because H is none empty, there exists $h \in H$,
By (III) $h^{-1} \in G$.
So, $1_G = h \circ h^{-1} \in H$.

So we have a triple $(H, \circ, 1_H)$, where $1_H = 1_G$
(I) Associativity
For every $h_{1,} h_2, h_3$, in H is satisfied because they are in G.
$h_1 \circ (h_2 \circ h_3) = (h_1 \circ h_2) \circ h_3$

(II) Identity
$1_H = 1_G$
Satisfies $1_H \circ h = h$
(III) Inverse
This is (III) in the previous proof.
This completes the proof.

Rotational symmetries:

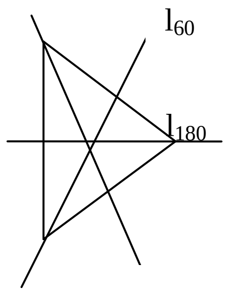


Triangle in R^2

Let R denote the rotation of R^2,
$R \times \mathbf{R}^2 \longrightarrow \mathbf{R}^2$

C_3 is in R, and
$C_3 = \mathrm{Stab}_R(\triangleright)$ is a subgroup.

Vertices at $(\cos(\Theta), \sin(\Theta))$
$\Theta = (0, 360/3, 360*2/3)$



Reflection with respect to these lines.
S_{l60} = Reflection with respect to l_{60}
S_{l180} = Reflection with respect to l_{180}
S_{l300} = Reflection with respect to l_{300}

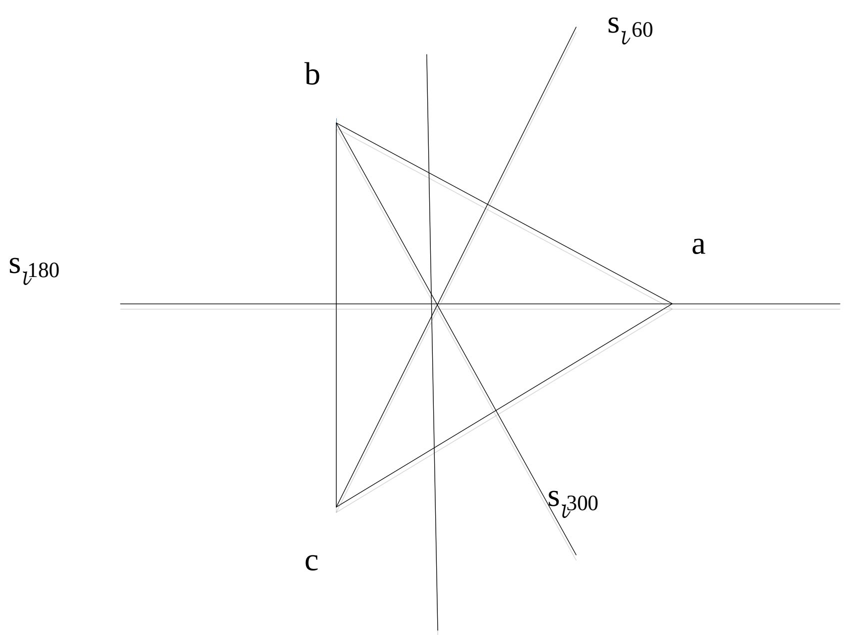
We obtain that the following collection of invertible linear operators:
$\{r_0, r_{120}, r_{240}, s_{l180}, s_{l60}, s_{l300}\}$

Claim: $\{r_0, r_{120}, r_{240}, s_{l180}, s_{l60}, s_{l300}\}$ is a group with respect to composition of operators.
Remark: We can check this directly.

02/18/2011

The symmetry group of the Triangle

Think of triangles with vertices at {(cosx,sinx) | x=0,120,240}
Think of the operation that preserves the triangle $\{r_0, r_{120}, r_{240}, s_{\ell 180}, s_{\ell 60}, s_{\ell 300}\}$



We may ask if they form a group with operation of product of matrices

$$s_{\ell 180}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

This is the reflection to X-axis

$$s_{\ell 180}\circ\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ -y\end{pmatrix}$$

Claim:

$$s_{\ell 60}= r_{120}\circ s_{\ell 180}$$

$$s_{\ell 300} = r_{240}\circ s_{\ell 180}$$

Proof:

Because $\{r_0, r_{120}, r_{240}, s_{l180}, s_{l60}, s_{l300}\}$ are linear operators acting on the basis of $\mathbf{R}^2$, if they act on the basis in the same way, then they are equal.

Since we first rotate vector **a** with 120 degree and then obtain its reflection about X-axis, we find that we successfully transform vector **a** into vector **b**.
$r_{120} \circ s_{l180}$ (a)=(b)
Also, we noticed that if we directly make vector **a** reflect about s_{l60} , we will transform **a** to **b**.

Similarly, we can find that

$s_{l300} = r_{240} \circ s_{l180}$

The proof is as simple as the proof of s_{l60}, the reader should verify this equation.

Now, we have got six elements in a set:

Denote $D=\{r_0, r_{120}, r_{240}, s_{l180}, s_{l60}, s_{l300}\}$

The question now arises as if they form a group with the operation of matrix multiplication. The answer is positive.

Claim: $(D, \circ, r_0)$ is a group.

Proof:

It is not hard to verity this claim by applying the definition of group, and we strongly encourage the reader verify the result this way themselves. But we want to introduce a more interesting but very important concept to proof the claim, orthogonal group.

What we will show then is that exist a group G acting on $\mathbf{R}^2$ s.t. $D=\{r_0, r_{120}, r_{240}, s_{l180}, s_{l60}, s_{l300}\}= \mathrm{Stab}_G(\mathbf{R}^2)$

The orthogonal group

We observed that all the six operators in D preserve the length of vectors and the angles up to a sign.

Recall that $\mathbf{R}^2$ has a natural inner product,

For each A, an operator in D

If we have (Au, Av) = (u , v)

We say A preserves the inner product.

Definition:

A matrix A in Matrix (2,**R**) is called <u>orthogonal</u> if it preserves the inner product i.e., (Au , Av)=(u , v) for any u , v in $\mathbf{R}^2$

Claim:

Denote O (2,**R**)={A|A is a 2x2 matrix s.t. A is orthogonal}

(1) {O (2,**R**), $\circ$, I}is a group, with operation of matrix multiplication.

Denote D_3= $Stab_{o(2)}(\triangleright)$

(2) D_3=D={r_0, r_{120}, r_{240}, $s_{\ell 180}$, $s_{\ell 60}$, $s_{\ell 300}$}

The proof of the claim is in the next lecture.