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Notes on the Self-Reducibility of the Weil Representation and Higher-Dimensional Quantum Chaos

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Summary. In these notes we discuss the *self-reducibility property* of the Weil representation. We explain how to use this property to obtain sharp estimates of certain higher-dimensional exponential sums which originate from the theory of quantum chaos. As a result, we obtain the Hecke quantum unique ergodicity theorem for a generic linear symplectomorphism A of the torus $\mathbb{T} = \mathbb{R}^{2N}/\mathbb{Z}^{2N}$.

Key words: Hannay–Berry Model, Quantum Unique Ergodicity, Bounds on Exponential Sums, Weil Representation, Self-Reducibility.

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1 Introduction

1.1 The Weil representation

In his celebrated 1964 Acta paper [34] Weil constructed a certain (projective) unitary representation of a symplectic group over a local field k (for example k could be $\mathbb{R}$, $\mathbb{C}$, or a p-adic field). This representation has many fascinating properties which have gradually been brought to light over the last few decades. It now appears that this representation is a central object, bridging various topics in mathematics and physics, including number theory, the theory of theta functions and automorphic forms, invariant theory, harmonic analysis, and quantum mechanics. Although it holds such a fundamental status, it is satisfying to observe that the Weil representation already appears in the study of functions on linear spaces. Given a k-linear space L, there exists an associated (polarized) symplectic vector space $V = L \times L^{*}$. The Weil representation of the group $Sp = Sp(V, \omega)$ can be realized on the Hilbert space

$\mathcal{H} = L^2(L, \mathbb{C})$. Interestingly, some elements of the group Sp act by certain kinds of generalized Fourier transforms. In particular, there exists a specific element $\mathrm{w} \in Sp$ (called the Weyl element) whose action is given, up to a normalization, by the standard Fourier transform. From this perspective, the classical theory of harmonic analysis seems to be devoted to the study of a particular operator in the Weil representation.

In these notes we will be concerned only with the case of the Weil representations of symplectic groups over finite fields. The main technical part is devoted to the study of a specific property of the Weil representation—the *self-reducibility property*. Briefly, this is a property concerning a relationship between the Weil representations of symplectic groups of different dimensions. In parts of these notes we devoted some effort to developing a general theory. In particular, the results concerning the self-reducibility property apply also to the Weil representation over local fields.

We use the self-reducibility property to bound certain higher-dimensional exponential sums which originate from the theory of quantum chaos, thereby obtaining a proof of one of the main statements in the field—the Hecke quantum unique ergodicity theorem for a generic linear symplectomorphism of the $2N$-dimensional torus.

1.2 Quantum chaos problem

One of the main motivational problems in quantum chaos is [2, 3, 26, 30] describing eigenstates

$$\widetilde{H}\Psi = \lambda\Psi, \ \ \Psi \in \mathcal{H},$$

of a chaotic Hamiltonian

$$\widetilde{H} = Op(H) : \mathcal{H} \to \mathcal{H},$$

where $\mathcal{H}$ is a Hilbert space. We deliberately use the notation $Op(H)$ to emphasize the fact that the quantum Hamiltonian $\widetilde{H}$ is a quantization of a classical Hamiltonian $H : M \to \mathbb{C}$, where M is a classical symplectic phase space (usually the cotangent bundle of a configuration space $M = T^*X$, in which case $\mathcal{H} = L^2(X)$). In general, describing Ψ is considered to be an extremely complicated problem. Nevertheless, for a few mathematical models of quantum mechanics rigorous results have been obtained. We shall proceed to describe one of these models.

Hannay–Berry model

In [18] Hannay and Berry explored a model for quantum mechanics on the two-dimensional symplectic torus $(\mathbb{T}, \omega)$. Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group $\Gamma \simeq SL_2(\mathbb{Z})$. One of their main motivations was to study the phenomenon of quantum chaos in this model [26, 28]. More precisely, they considered an ergodic discrete dynamical system on the torus which is generated by a hyperbolic automorphism $A \in \Gamma$. Quantizing the system, the classical phase space $(\mathbb{T}, \omega)$ is replaced by a *finite dimensional* Hilbert space $\mathcal{H}$, classical observables, i.e., functions $f \in C^{\infty}(\mathbb{T})$, by operators $\pi(f) \in \mathrm{End}(\mathcal{H})$, and classical symmetries by a unitary representation $\rho : \Gamma \to U(\mathcal{H})$.

Shnirelman's theorem

Analogous with the case of the Schrödinger equation, consider the following eigenstates problem

$$\rho(A)\Psi = \lambda\Psi.$$

A fundamental result, valid for a wide class of quantum systems which are associated to ergodic classical dynamics, is Shnirelman's theorem [31], asserting that in the semi-classical limit almost all (in a suitable sense) eigenstates become equidistributed in an appropriate sense.

A variant of Shnirelman's theorem also holds in our situation [4]. More precisely, we have that in the semi-classical limit $\hbar \to 0$ for almost all (in a suitable sense) eigenstates Ψ of the operator $\rho(A)$ the corresponding *Wigner distribution* $\langle \Psi | \pi(\cdot) \Psi \rangle : C^{\infty}(\mathbb{T}) \to \mathbb{C}$ approaches the phase space average $\int_{\mathbb{T}} \cdot |\omega|$. In this respect, it seems natural to ask whether there exist exceptional sequences of eigenstates? Namely, eigenstates that do not obey the Shnirelman's rule (*scarred* eigenstates). It was predicted by Berry [2, 3] that *scarring* phenomenon is not expected to be seen for quantum systems associated with *generic* chaotic classical dynamics. However, in our situation the operator $\rho(A)$ is not generic, and exceptional eigenstates were constructed. Indeed, it was confirmed mathematically in [8] that certain $\rho(A)$-eigenstates might localize. For example, in that paper a sequence of eigenstates Ψ was constructed, for which the corresponding Wigner distribution approaches the measure $\frac{1}{2}\delta_0 + \frac{1}{2}|\omega|$ on $\mathbb{T}$.

Hecke quantum unique ergodicity

A quantum system that obeys Shnirelman's rule is also called quantum ergodic. Can one impose some natural conditions on the eigenstates so that no exceptional eigenstates will appear? Namely, *quantum unique ergodicity* will hold. This question was addressed in a paper by Kurlberg and Rudnick [25]. In that paper, they formulated a rigorous notion of Hecke quantum unique ergodicity for the case $\hbar = 1/p$. The following is a brief description of that work. The basic observation is that the degeneracies of the operator $\rho(A)$ are coupled with the existence of symmetries. There exists a commutative group of operators that commutes with $\rho(A)$, which can in fact be computed. In more detail, the representation ρ factors through the quotient group $Sp = SL_2(\mathbb{F}_p)$. We denote by $T_A \subset Sp$ the centralizer of the element A, now considered as an element of the quotient group. The group T_A is called (cf. [25]) the *Hecke torus* corresponding to the element A. The Hecke torus acts semisimply on $\mathcal{H}$. Therefore, we have a decomposition

$$\mathcal{H} = \bigoplus_{\chi : T_A \to \mathbb{C}^\times} \mathcal{H}_\chi,$$

where $\mathcal{H}_\chi$ is the Hecke eigenspace corresponding to the character χ. Consider a unit eigenstate $\Psi \in \mathcal{H}_\chi$ and the corresponding Wigner distribution $\mathcal{W}_\chi : C^\infty(\mathbb{T}) \to \mathbb{C}$, defined by the formula $\mathcal{W}_\chi(f) = \langle \Psi | \pi(f) \Psi \rangle$. The main statement in [25] proves an explicit bound on the semi-classical asymptotic of $\mathcal{W}_\chi(f)$

$$\left| \mathcal{W}_\chi(f) - \int_{\mathbb{T}} f|\omega| \right| \leq \frac{C_f}{p^{1/4}},$$

where C_f is a constant that depends only on the function f. In Rudnick's lectures at MSRI, Berkeley 1999 [27], and ECM, Barcelona 2000 [28], he conjectured that a stronger bound should hold true, i.e.,

$$\left| \mathcal{W}_\chi(f) - \int_{\mathbb{T}} f|\omega| \right| \leq \frac{C_f}{p^{1/2}}. \tag{1}$$

A particular case (which implies (1)) of the above inequality is when $f = \xi$, where ξ is a non-trivial character. In this case, the integral $\int_{\mathbb{T}} \xi|\omega|$ vanishes and in addition it turns out that $C_\xi = 2 + o(1)$. Hence, we obtain the following simplified form of (1)

$$|\mathcal{W}_\chi(\xi)| \leq \frac{2 + o(1)}{\sqrt{p}}, \tag{2}$$

for sufficiently large p. These stronger bounds were proved in the paper [13]. It will be instructive to briefly recall the main ideas and techniques used in [13].

Geometric approach

The basic observation to be made is that the theory of quantum mechanics on the torus, in the case $\hbar = 1/p$, can be equivalently recast in the language of the representation theory of finite groups in characteristic p. We will endeavor to give a more precise explanation of this matter. Consider the quotient $\mathbb{F}_p$-vector space $V = \mathbb{T}^\vee / p\mathbb{T}^\vee$, where $\mathbb{T}^\vee \simeq \mathbb{Z}^2$ is the lattice of characters on $\mathbb{T}$. We denote by $H = H(V)$ the Heisenberg group associated to V. The group Sp is naturally identified with the group of linear symplectomorphisms of V. We have an action of Sp on H. The Stone–von Neumann theorem (see Theorem 5) states that there exists a unique irreducible representation $\pi : H \to GL(\mathcal{H})$, with a non-trivial character ψ of the center of H. As a consequence of its uniqueness, its isomorphism class is fixed by Sp. This is equivalent to saying that $\mathcal{H}$ is equipped with a compatible projective representation $\rho : Sp \to PGL(\mathcal{H})$, which in fact can be linearized to an honest representation. This representation is the celebrated Weil representation. Noting that Sp is the group of rational points of the algebraic group **Sp** (we use boldface letters to denote algebraic varieties), it is natural to ask whether there exists an algebro-geometric object that underlies the representation ρ. The answer to this question is positive. The construction is proposed in an unpublished letter of Deligne to Kazhdan [7], which appears now in [13, 16]. Briefly, the content of this letter is a construction of *representation sheaf* $\mathcal{K}_\rho$ on the algebraic variety **Sp**. We obtain, as a consequence, the following general principle:

Motivic principle. *All quantum mechanical quantities in the Hannay–Berry model are motivic in nature.*

By this we mean that every quantum-mechanical quantity $\mathcal{Q}$ is associated with a vector space $V_{\mathcal{Q}}$ (certain cohomology of a suitable ℓ-adic sheaf) endowed with a Frobenius action $\mathrm{Fr} : V_{\mathcal{Q}} \to V_{\mathcal{Q}}$ so that $\mathcal{Q} = \mathrm{Tr}(\mathrm{Fr}_{|V_{\mathcal{Q}}})$. In particular, it was shown in [13] that there exists a two-dimensional vector space V_χ, endowed with an action $\mathrm{Fr} : V_\chi \to V_\chi$, so that

$$\mathcal{W}_\chi(\xi) = \mathrm{Tr}(\mathrm{Fr}_{|V_\chi}). \tag{3}$$

This, combined with the purity condition that the eigenvalues of Fr are of absolute value $1/\sqrt{p}$, implies the estimate (2).

The higher-dimensional Hannay–Berry model

The higher-dimensional Hannay–Berry model is obtained as a quantization of a $2N$-dimensional symplectic torus $(\mathbb{T}, \omega)$ acted upon by the group $\Gamma \simeq Sp(2N, \mathbb{Z})$ of linear symplectic automorphisms. It was first constructed in [12], where, in particular, a quantization of the whole group of symmetries Γ was obtained. Consider a regular ergodic element $A \in \Gamma$, i.e., A generates an ergodic discrete dynamical system and it is regular in the sense that it has distinct eigenvalues over $\mathbb{C}$. It is natural to ask whether quantum unique ergodicity will hold true in this setting as well, as long as one takes into account the whole group of hidden (Hecke) symmetries? Interestingly, the answer to this question is NO! Several new results in this direction have been announced recently. In the case where the automorphism A is *non-generic*, meaning that it has an invariant Lagrangian (and more generally co-isotropic) sub-torus $\mathbb{T}_L \subset \mathbb{T}$, an interesting new phenomenon was revealed. There exists a sequence $\{\Psi_\hbar\}$ of Hecke eigenstates which are closely related to the physical phenomena of *localization*, known in the physics literature (cf. [20, 24]) as *scars*. We will call them *Hecke scars*. These states are localized in the sense that the associated Wigner distribution converges to the Haar measure μ on the invariant Lagrangian sub-torus

$$\mathcal{W}_{\Psi_\hbar}(f) \to \int\limits_{\mathbb{T}_L} f d\mu, \text{ as } \hbar \to 0, \tag{4}$$

for every smooth observable f. These special kinds of Hecke eigenstates were first established in [10]. The semi-classical interpretation of the localization phenomena (4) was announced in [23].

The above phenomenon motivates the following definition:

Definition 1. *An element $A \in \Gamma$ is called generic if it is regular and admits no non-trivial invariant co-isotropic sub-tori.*

Remark 1. The collection of generic elements constitutes an open subscheme of Γ. In particular, a generic element need not be ergodic automorphism of $\mathbb{T}$. However, in the case where $\Gamma \simeq SL_2(\mathbb{Z})$ every ergodic (i.e., hyperbolic) element is generic. An example of a generic element which is not ergodic is given by the Weyl element $\mathrm{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In these notes we will require the automorphism $A \in \Gamma$ to be generic. This case was first considered in [14], where using similar geometric techniques as in [13] the analogue of inequality (2) was obtained. For the sake of simplicity, let us assume that the automorphism A is *strongly generic*, i.e., it has no non-trivial invariant sub-tori.

Theorem 1 ([14]). *Let ξ be a non-trivial character of $\mathbb{T}$. The following bound holds*

$$|\mathcal{W}_\chi(\xi)| \leq \frac{[2+o(1)]^N}{\sqrt{p}^N}, \tag{5}$$

where p is a sufficiently large prime number.

In particular, using the bound (5), we obtain the following statement for general observable:

Corollary 1 (Hecke quantum unique ergodicity). *Consider an observable $f \in C^\infty(\mathbb{T})$ and a sufficiently large prime number p. Then*

$$\left| \mathcal{W}_\chi(f) - \int\limits_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^N},$$

where $\mu = |\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

In these notes, using the self-reducibility property of the Weil representation, we improve the above estimates and obtain the following theorem:

Theorem 2 (Sharp bound). *Let ξ be a non-trivial character of $\mathbb{T}$. For sufficiently large prime number p we have*

$$|\mathcal{W}_\chi(\xi)| \leq \frac{[2+o(1)]^{r_p}}{\sqrt{p}^N}, \tag{6}$$

where the number r_p is an integer between 1 and N, that we will call the symplectic rank of T_A.

Remark 2. It will be shown (see Subsection 6.2) that the distribution of the symplectic rank r_p (6) in the set $\{1, \ldots, N\}$ is governed by the Chebotarev density theorem applied to a suitable Galois group. For example, in the case where $A \in Sp(4, \mathbb{Z})$ is strongly generic we have

$$\lim_{x \to \infty} \frac{\#\{r_p = r \mid p \leq x\}}{\pi(x)} = \tfrac{1}{2}, \qquad r = 1,\ 2,$$

where $\pi(x)$ denotes the number of primes up to x.

Remark 3. For the more general version of Theorem 2, one that holds in the general generic case (Definition 1), see Subsection 6.3.

In order to witness the improvement of (6) over (5), it would be instructive to consider the following extreme scenario. Assume that the Hecke torus T_A acts on $V \simeq \mathbb{F}_p^{2N}$ irreducibly. In this case it turns out that $r_p = 1$. Hence, (6) becomes

$$|\mathcal{W}_\chi(\xi)| \leq \frac{2+o(1)}{\sqrt{p}^N},$$

which constitutes a significant improvement over the coarse topological estimate (5). Let us elaborate on this. Recall the motivic interpretation (3) of the Wigner distribution. In [14] an analogous interpretation was given to the higher-dimensional Wigner distributions, realizing them as $\mathcal{W}_\chi(\xi) = \mathrm{Tr}(\mathrm{Fr}_{|V_\chi})$, where, by the purity condition, the eigenvalues of Fr are of absolute value $1/\sqrt{p}^N$. But, in this setting the dimension of V_χ is not 2, but 2^N, i.e., the Frobenius looks like

$$\mathrm{Fr} = \begin{pmatrix} \lambda_1 & * & * & * \\ & \cdot & * & * \\ & & \cdot & * \\ & & & \lambda_{2^N} \end{pmatrix}.$$

Hence, if we use only this amount of information, then the best estimate which can be obtained is (5). Therefore, in this respect the problem that we confront is showing cancellations between different eigenvalues, more precisely angles, of the Frobenius operator acting on a high-dimensional vector space, i.e., cancellations in the sum $\sum_{j=1}^{2^N} e^{i\theta_j}$, where the angles $0 \leq \theta_j < 2\pi$ are defined via $\lambda_j = e^{i\theta_j}/\sqrt{p}^N$. This problem is of a completely different nature, which is not accounted for by standard cohomological techniques (we thank R. Heath-Brown for pointing out to us [19] about the phenomenon of cancellations between Frobenius eigenvalues in the presence of high-dimensional cohomologies).

Remark 4. Choosing a realization $\mathcal{H} \simeq \mathbb{C}(\mathbb{F}_p^N)$, the matrix coefficient $\mathcal{W}_\chi(\xi)$ is equivalent to an exponential sum of the form

$$\langle \Psi | \pi(\xi) \Psi \rangle = \sum_{x \in \mathbb{F}_p^N} \Psi(x) e^{\frac{2\pi i}{p} \xi_+ x} \overline{\Psi}(x + \xi_-). \tag{7}$$

Here one encounters two problems. First, it is not so easy to describe the eigenstates Ψ. Second, the sum (7) is a high-dimensional exponential sum (over $\mathbb{F}_p$), which is known to be hard to analyze using standard techniques. The crucial point that we explain in these notes is that it can be realized, essentially, as a one-dimensional exponential sum over $\mathbb{F}_q$, where $q = p^N$.

1.3 Solution via self-reducibility

Let us explain the main idea underlying the proof of estimate (6). Let us assume for the sake of simplicity that the Hecke torus is completely inert, i.e., acts irreducibly on the vector space $V \simeq \mathbb{F}_p^{2N}$.

Representation theoretic interpretation of the Wigner distribution

The Hecke eigenstate Ψ is a vector in a representation space $\mathcal{H}$. The space $\mathcal{H}$ supports the Weil representation of the symplectic group $Sp \simeq Sp(2N, k)$, $k = \mathbb{F}_p$. The vector Ψ is completely characterized in representation theoretic terms, as being a character vector of the Hecke torus T_A. As a consequence, all quantities associated to Ψ, and in particular the Wigner distribution $\mathcal{W}_\chi$ are characterized in terms of the Weil representation. The main observation to be made is that the Hecke state Ψ can be characterized in terms of another Weil representation, this time of a group of much smaller dimension. In fact, it can be characterized, roughly, in terms of the Weil representation of $SL_2(K)$, $K = \mathbb{F}_{p^N}$.

Self-reducibility property

A fundamental notion in our study is that of a *symplectic module structure*. A symplectic module structure is a triple $(K, V, \overline{\omega})$, where K is a finite dimensional commutative algebra over k, equipped with an action on the vector space V, and $\overline{\omega}$ is a K-linear symplectic form satisfying the property $\mathrm{Tr}_{K/k}(\overline{\omega}) = \omega$. Let us assume for the sake of simplicity that K is a field. Let

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$\overline{Sp} = Sp(V, \overline{\omega})$ be the group of K-linear symplectomorphisms with respect to the form $\overline{\omega}$. There exists a canonical embedding

$$\iota : \overline{Sp} \hookrightarrow Sp. \tag{8}$$

We will be mainly concerned with symplectic module structures which are associated to maximal tori in Sp. More precisely, it will be shown that associated to a maximal torus $T \subset Sp$ there exists a canonical symplectic module structure $(K, V, \overline{\omega})$ so that $T \subset \overline{Sp}$. The most extreme situation is when the torus $T \subset Sp$ is completely inert, i.e., acts irreducibly on the vector space V. In this particular case, the algebra K is in fact a field with $\dim_K V = 2$ which implies that $\overline{Sp} \simeq SL_2(K)$, i.e., using (8) we get $T \subset SL_2(K) \subset Sp$.

Let us denote by $(\rho, Sp, \mathcal{H})$ the Weil representation of Sp. The main observation now is (cf. [9]) the following:

Theorem 3 (Self-reducibility property). *The restricted representation* $(\overline{\rho} = \iota^* \rho, SL_2(K), \mathcal{H})$ *is the Weil representation of* $SL_2(K)$.

Applying the self-reducibility property to the Hecke torus T_A, it follows that the Hecke eigenstates Ψ can be characterized in terms of the Weil representation of $SL_2(K)$. Therefore, in this respect, Theorem 2 is reduced to the result obtained in [13].

1.4 Quantum unique ergodicity for statistical states

Let $A \in \Gamma$ be a generic linear symplectomorphism. As in harmonic analysis, one would like to use Theorem 2 concerning the Hecke eigenstates in order to extract information on the spectral theory of the operator $\rho(A)$ itself. For the sake of simplicity, let us assume that A is strongly generic, i.e., it acts on the torus $\mathbb{T}$ with no non-trivial invariant sub-tori. Next, a possible reformulation of the quantum unique ergodicity statement, one which is formulated for the automorphism A itself instead of the Hecke group of symmetries, is presented.

The element A acts via the Weil representation ρ on the space $\mathcal{H}$ and decomposes it into a direct sum of $\rho(A)$-eigenspaces

$$\mathcal{H} = \bigoplus_{\lambda \in Spec(\rho(A))} \mathcal{H}_\lambda. \tag{9}$$

Considering an $\rho(A)$-eigenstate Ψ and the corresponding projector P_Ψ one usually studies the Wigner distribution $\langle \Psi | \pi(\xi) \Psi \rangle = \mathrm{Tr}(\pi(\xi) P_\Psi)$ which, due to the fact that $rank(P_\Psi) = 1$, is sometimes called *pure state*. In the same way, we might think about an Hecke–Wigner distribution $\langle \Psi | \pi(\xi) \Psi \rangle = \mathrm{Tr}(\pi(\xi) P_\chi)$, attached to a T_A-eigenstate Ψ, as a *pure Hecke state*. Following von Neumann [33] we suggest the possibility of looking at the more general *statistical state*, defined by a non-negative, self-adjoint operator D, called the von Neumann density operator, normalized to have $\mathrm{Tr}(D) = 1$. For example, to the automorphism A we can attach the natural family of density operators $D_\lambda = \frac{1}{m_\lambda} P_\lambda$, where P_λ is the projector on the eigenspace $\mathcal{H}_\lambda$ (9), and $m_\lambda = \dim(\mathcal{H}_\lambda)$. Consequently, we obtain a family of statistical states

$$\mathcal{W}_\lambda(\cdot) = \mathrm{Tr}(\pi(\cdot) D_\lambda).$$

Theorem 4. *Let ξ be a non-trivial character of $\mathbb{T}$. For a sufficiently large prime number p, and every statistical state $\mathcal{W}_\lambda$, we have*

$$|\mathcal{W}_\lambda(\xi)| \leq \frac{(2+o(1))^{r_p}}{\sqrt{p}^N}, \qquad (10)$$

where r_p is an explicit integer $1 \leq r_p \leq N$ which is determined by A.

Theorem 4 follows from the fact that the Hecke torus T_A acts on the spaces $\mathcal{H}_\lambda$, and hence, one can use the Hecke eigenstates, and the bound (6). In particular, using (10) we obtain for a general observable the following bound:

Corollary 2. *Consider an observable $f \in C^\infty(\mathbb{T})$ and a sufficiently large prime number p. Then*

$$\left| \mathcal{W}_\lambda(f) - \int_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^N},$$

where $\mu = |\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

1.5 Results

1. *Bounds of higher-dimensional exponential sums.* The main results of these notes are a sharp estimates of certain higher-dimensional exponential sums attached to tori in $Sp(2N, \mathbb{F}_q)$. This is the content of Theorems 12 and 14 and is obtained using the self-reducibility property of the Weil representation as stated in Theorems 9 and 10.
2. *Hecke quantum unique ergodicity theorem.* The main application of these notes is the proof of the *Hecke quantum unique ergodicity theorem*, i.e., Theorems 17 and 18, for generic linear symplectomorphism of the torus in any dimension. The proof of the theorem is a direct application of the sharp bound on the higher-dimensional exponential sums.
3. *Multiplicities formula.* Exact formula for the multiplicities, i.e., the dimensions of the character spaces for the action of maximal tori in the Weil representation are derived. This is obtained first for the $SL_2(\mathbb{F}_q)$ case in Theorem 8 using the character formula presented in Theorem 7. Then, as a direct application of the self-reducibility property, the formula is extended in Theorem 11 to the higher-dimensional cases.

In addition, a formulation of the quantum unique ergodicity statement for quantum chaos problems, close in spirit to the von Neumann idea about density operator, is suggested in Theorem 4. The statement includes only the quantum operator A rather than the whole Hecke group of symmetries [25]. The proof of the statement uses the Hecke operators as a harmonic analysis tool.

1.6 Structure of the notes

Apart from the introduction, the notes consist of five sections.

In Section 2 we give some preliminaries on representation theory which are used in the notes. In Subsection 2.3 we recall the invariant presentation of the Weil representation over finite fields [16], and we discuss applications to multiplicities. Section 3 constitutes the main technical part of this work. Here we develop the theory that underlies the self-reducibility property of the Weil representation. In particular, in Subsection 3.1 we introduce the notion of symplectic module structure. In Subsection 3.2 we prove the existence of symplectic module structure associated with a maximal torus in Sp. Finally, we establish the self-reducibility property of the Weil representation, i.e., Theorem 10, and apply this property to get information on multiplicities in Subsection 3.4. Section 4 is devoted to an application of the theory developed in previous sections to estimating higher-dimensional exponential sums which originate from the mathematical theory of quantum chaos. In Section 5 we describe the higher-dimensional Hannay–Berry model of quantum mechanics on the torus. Finally, in Section 6 we present the main application of these notes—the proof of the Hecke quantum unique ergodicity theorem for generic linear symplectomorphisms of the $2N$-dimensional torus.

Remark 5. Complete proofs for the statements appearing in these notes will be given elsewhere.

1.7 Acknowledgments

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2 Preliminaries

In this section, we denote by $k = \mathbb{F}_q$ the finite field of q elements and odd characteristic.

2.1 The Heisenberg representation

Let (V, ω) be a $2N$-dimensional symplectic vector space over the finite field k. There exists a two-step nilpotent group $H = H(V, \omega)$ associated to the symplectic vector space (V, ω). The group H is called the *Heisenberg group*. It can be realized as the set $H = V \times k$, equipped with the multiplication rule

$$(v, z) \cdot (v', z') = (v + v', z + z' + \tfrac{1}{2}\omega(v, v')).$$

The center of H is $Z(H) = \{(0, z) : z \in k\}$. Fix a non-trivial central character $\psi : Z(H) \longrightarrow \mathbb{C}^{\times}$. We have the following fundamental theorem:

Theorem 5 (Stone–von Neumann). *There exists a unique (up to isomorphism) irreducible representation $(\pi, H, \mathcal{H})$ with central character ψ, i.e., $\pi(z) = \psi(z) Id_{\mathcal{H}}$ for every $z \in Z(H)$.*

We call the representation π appearing in Theorem 5, the *Heisenberg representation* associated with the central character ψ.

Remark 6. The representation π, although it is unique, admits a multitude of different models (realizations). In fact, this is one of its most interesting and powerful attributes. In particular, to any Lagrangian splitting $V = L' \oplus L$, there exists a model $(\pi_{L',L}, H, \mathbb{C}(L))$, where $\mathbb{C}(L)$ denotes the space of complex valued functions on L. In this model, we have the following actions:

- $\pi_{L',L}(l')[f](x) = \psi(\omega(l', x)) f(x)$;
- $\pi_{L',L}(l)[f](x) = f(x + l)$;
- $\pi_{L',L}(z)[f](x) = \psi(z) f(x)$,
  where $l' \in L'$, $x, l \in L$, and $z \in Z(H)$.

The above model is called the Schrödinger realization associated with the splitting $V = L' \oplus L$.

2.2 The Weyl transform

Given a linear operator $A : \mathcal{H} \to \mathcal{H}$ we can associate to it a function on the group H defined as follows

$$W(A)(h) = \tfrac{1}{\dim \mathcal{H}} \mathrm{Tr}(A\pi(h^{-1})). \tag{11}$$

The transform $W : \mathrm{End}(\mathcal{H}) \to \mathbb{C}(H)$ is called the *Weyl transform [21, 35]*. The Weyl transform admits a left inverse $\pi : \mathbb{C}(H) \to \mathrm{End}(\mathcal{H})$ given by the extended action $\pi(K) = \sum\limits_{h \in H} K(h)\pi(h)$.

2.3 The Weil representation

Let $Sp = Sp(V, \omega)$ denote the group of linear symplectic automorphisms of V. The group Sp acts by group automorphisms on the Heisenberg group through its tautological action on the vector space V. A direct consequence of Theorem 5 is the existence of a projective representation $\widetilde{\rho} : Sp \to PGL(\mathcal{H})$. The classical construction of $\widetilde{\rho}$ out of the Heisenberg representation π is due to Weil [34]. Considering the Heisenberg representation π and an element $g \in Sp$, one can define a new representation π^g acting on the same Hilbert space via $\pi^g(h) = \pi(g(h))$. Clearly both π and π^g have central character ψ; hence, by Theorem 5, they are isomorphic. Since the space $\mathsf{Hom}_H(\pi, \pi^g)$ is one-dimensional, choosing for every $g \in Sp$ a non-zero representative $\widetilde{\rho}(g) \in \mathsf{Hom}_H(\pi, \pi^g)$ gives the required projective representation. In more concrete terms, the projective representation $\widetilde{\rho}$ is characterized by the formula

$$\widetilde{\rho}(g)\, \pi(h)\, \widetilde{\rho}\left(g^{-1}\right) = \pi(g(h)), \tag{12}$$

for every $g \in Sp$ and $h \in H$. It is a peculiar phenomenon of the finite field setting that the projective representation $\widetilde{\rho}$ can be linearized into an honest representation. This linearization is unique, except in the case the finite field is $\mathbb{F}_3$ and $\dim V = 2$ (for the canonical choice in the latter case see [17]).

Theorem 6. *There exists a canonical unitary representation*

$$\rho : Sp \longrightarrow GL(\mathcal{H}),$$

satisfying the formula (12).

Invariant presentation of the Weil representation

An elegant description of the Weil representation can be obtained [16] using the Weyl transform (see Subsection 2.2). Given an element $g \in Sp$, the operator $\rho(g)$ can be written as $\rho(g) = \pi(K_g)$, where K_g is the Weyl transform $K_g = W(\rho(g))$. The homomorphism property of ρ is manifested as

$$K_g * K_h = K_{gh} \quad \text{for every} \quad g, h \in Sp,$$

where $*$ denotes (properly normalized) group theoretic convolution on H. Finally, the function K can be explicitly described on an appropriate subset of Sp [16]. Let $U \subset Sp$ denote the subset consisting of all elements $g \in Sp$ such that $g - I$ is invertible. For every $g \in U$ and $v \in V$ we have

$$K_g(v) = \nu(g)\psi(\tfrac{1}{4}\omega(\kappa(g)v, v)), \tag{13}$$

where $\kappa(g) = \frac{g+I}{g-I}$ is the Cayley transform [21, 36], and

$$\nu(g) = (G/q)^{2N}\sigma(\det(g - I)),$$

with σ the unique quadratic character of the multiplicative group $\mathbb{F}_q^{\times}$, and $G = \sum\limits_{z \in Z(H)} \psi(z^2)$ the quadratic Gauss sum.

2.4 The Heisenberg–Weil representation

Let J denote the semi-direct product $J = Sp \ltimes H$. The group J is sometimes referred to as the *Jacobi* group. The compatible pair (π, ρ) is equivalent to a single representation $\tau : J \longrightarrow GL(\mathcal{H})$ of the Jacobi group defined by the formula $\tau(g, h) = \rho(g)\pi(h)$. It is an easy exercise to verify that the Egorov identity (12) implies the multiplicativity of the map τ.

In these notes, we would like to adopt the name *Heisenberg–Weil representation* when referring to the representation τ.

2.5 Character formulas

The invariant presentation (11) and formula (13) imply [16] a formula for the character of the $2N$-dimensional Heisenberg–Weil representation over a finite field (cf. [9, 22]).

Theorem 7 (Character formulas [16]). *The character ch_ρ of the Weil representation, when restricted to the subset U, is given by*

$$ch_\rho(g) = \sigma((-1)^N \det(g - I)), \tag{14}$$

and the character ch_τ of the Heisenberg–Weil representation, when restricted to the subset $U \times H$, is given by

$$\mathrm{ch}_\tau(g, v, z) = \mathrm{ch}_\rho(g)\psi(\tfrac{1}{4}\omega(\kappa(g)v, v) + z). \tag{15}$$

2.6 Application to multiplicities

We would like to apply the formula (15) to the study of the multiplicities arising from actions of tori via the Weil representation (cf. [1, 9, 32]). Let us start with the two-dimensional case (see Theorem 11 for the general case). Let $T \subset Sp \simeq SL_2(\mathbb{F}_q)$ be a maximal torus. The torus T acts semisimply on $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \bigoplus\limits_{\chi:T\to\mathbb{C}^\times} \mathcal{H}_\chi$.

As a consequence of having the explicit formula (14), we obtain a simple description for the multiplicities $m_\chi = \dim \mathcal{H}_\chi$. Denote by $\sigma : T \to \mathbb{C}^\times$ the unique quadratic character of T.

Theorem 8 (Multiplicities formula). *We have $m_\chi = 1$ for any character $\chi \neq \sigma$. Moreover, $m_\sigma = 2$ or 0, depending on whether the torus T is split or inert, respectively.*

What about the multiplicities for action of tori in the Weil representation of higher-dimensional symplectic groups? This problem can be answered (see Theorem 11) using the self-reducibility property of the Weil representation.

3 Self-reducibility of the Weil representation

In this section, unless stated otherwise, the field k is an *arbitrary field* of characteristic different from two.

3.1 Symplectic module structures

Let K be a finite-dimensional commutative algebra over the field k. Let $\mathrm{Tr} : K \to k$ be the trace map, associating to an element $x \in K$ the trace of the k-linear operator $m_x : K \to K$ obtained by left multiplication by the element x. Consider a symplectic vector space (V, ω) over k.

Definition 2. *A symplectic K-module structure on (V, ω) is an action $K \otimes_k V \to V$, and a K-linear symplectic form $\overline{\omega} : V \times V \to K$ such that*

$$Tr \circ \overline{\omega} = \omega. \tag{16}$$

Given a symplectic module structure $(K, V, \overline{\omega})$ on a symplectic vector space (V, ω), we denote by $\overline{Sp} = Sp(V, \overline{\omega})$ the group of K-linear symplectomorphisms with respect to the form $\overline{\omega}$. The compatibility condition (16) gives a natural embedding

$$\iota : \overline{Sp} \hookrightarrow Sp. \tag{17}$$

3.2 Symplectic module structure associated with a maximal torus

Let $T \subset Sp$ be a maximal torus.

A particular case

In order to simplify the presentation, let us assume first that T acts irreducibly on the vector space V, i.e., there exists no non-trivial T-invariant subspaces. Let $A = Z(T, \mathrm{End}(V))$, be the centralizer of T in the algebra of all linear endomorphisms. Clearly (due to the assumption of irreducibility) A is a division algebra. Moreover, we have

Claim. The algebra A is commutative.

In particular, this claim implies that A is a field extension of k. Let us now describe a special quadratic element in the Galois group $\mathrm{Gal}(A/k)$. Denote by $(\cdot)^t : \mathrm{End}(V) \to \mathrm{End}(V)$ the symplectic transpose characterized by the property

$$\omega(Rv, u) = \omega(v, R^t u),$$

for all $v, u \in V$, and every $R \in \mathrm{End}(V)$. It can be easily verified that $(\cdot)^t$ preserves A, leaving the subfield k fixed, hence, it defines an element $\Theta \in \mathrm{Gal}(A/k)$, satisfying $\Theta^2 = \mathrm{Id}$. Denote by $K = A^{\Theta}$ the subfield of A consisting of elements fixed by Θ. We have the following proposition:

Proposition 1 (Hilbert's Theorem 90). *We have* $\dim_K V = 2$.

Corollary 3. *We have* $\dim_K A = 2$.

As a corollary, we have the following description of T. Denote by $N_{A/K} : A \to K$ the standard norm map.

Corollary 4. *We have* $T = S(A) = \left\{a \in A : N_{A/K}(a) = 1\right\}$

The symplectic form ω can be lifted to a K-linear symplectic form $\overline{\omega}$, which is invariant under the action of the torus T. This is the content of the following proposition:

Proposition 2 (Existence of canonical symplectic module structure). *There exists a canonical T-invariant K-linear symplectic form $\overline{\omega} : V \times V \to K$ satisfying the property $Tr \circ \overline{\omega} = \omega$.*

Concluding, we obtained a T-invariant symplectic K-module structure on V.

Let $\overline{Sp} = Sp(V, \overline{\omega})$ denote the group of K-linear symplectomorphisms with respect to the symplectic form $\overline{\omega}$. We have (17) a natural embedding $\overline{Sp} \subset Sp$. The elements of T commute with the action of K, and preserve the symplectic form $\overline{\omega}$ (Proposition 2); hence, we can consider T as a subgroup of $\overline{Sp}$. By Proposition 1 we can identify $\overline{Sp} \simeq SL_2(K)$, and using (17) we obtain

$$T \subset SL_2(K) \subset Sp. \tag{18}$$

To conclude we see that T consists of the K-rational points of a maximal torus $\mathbf{T} \subset \mathbf{SL}_2$ (in this case T consists of the rational points of an inert torus).

General case

Here, we drop the assumption that T acts irreducibly on V. By the same argument as before, the algebra $A = Z(T, \mathrm{End}(V))$ is commutative, yet, it may no longer be a field. The symplectic transpose $(\cdot)^t$ preserves the algebra A, and induces an involution $\Theta : A \to A$. Let $K = A^{\Theta}$ be the subalgebra consisting of elements $a \in A$ fixed by Θ. Following the same argument as in the proof of Proposition 1, we can show that V is a free K-module of rank 2. Following the same arguments as in the proof of Proposition 2, we can show that there exists a canonical symplectic form $\overline{\omega} : V \times V \to K$, which is K-linear and invariant under the action of the torus T. Concluding, associated to a maximal torus T there exists a T-invariant symplectic K-module structure

$$(K, V, \overline{\omega}). \tag{19}$$

Denote by $\overline{Sp} = Sp(V, \overline{\omega})$ the group of K-linear symplectomorphisms with respect to the form $\overline{\omega}$. We have a natural embedding

$$\iota_S : \overline{Sp} \hookrightarrow Sp \tag{20}$$

and we can consider T as a subgroup of $\overline{Sp}$. Finally, we have $\overline{Sp} \simeq SL_2(K)$, and T consists of the K-rational points of a maximal torus $\mathbf{T} \subset \mathbf{SL}_2$. In particular, the relation (18) holds also in this case: $T \subset SL_2(K) \subset Sp$.

We shall now proceed to give a finer description of all objects discussed so far. The main technical result is summarized in the following lemma:

Lemma 1 (Symplectic decomposition). *We have a canonical decomposition*

$$(V, \omega) = \bigoplus_{\alpha \in \Xi} (V_\alpha, \omega_\alpha), \tag{21}$$

into (T, A)-invariant symplectic subspaces. In addition, we have the following associated canonical decompositions

1. $T = \prod T_\alpha$, where T_α consists of elements $t \in T$ such that $t_{|V_\beta} = \mathrm{Id}$ for every $\beta \neq \alpha$.
2. $A = \bigoplus A_\alpha$, where A_α consists of elements $a \in A$ such that $a_{|V_\beta} = \mathrm{Id}$ for every $\beta \neq \alpha$. Moreover, each sub-algebra A_α is preserved under the involution Θ.
3. $K = \bigoplus K_\alpha$, where $K_\alpha = A_\alpha^\Theta$. Moreover, K_α is a field and $\dim_{K_\alpha} V_\alpha = 2$.
4. $\varpi = \bigoplus \varpi_\alpha$, where $\varpi_\alpha : V_\alpha \times V_\alpha \to K_\alpha$ is a K_α-linear T_α-invariant symplectic form satisfying $\mathrm{Tr} \circ \varpi_\alpha = \omega_\alpha$.

Definition 3. *We will call the set Ξ (21) the symplectic type of T and the number $|\Xi|$ the symplectic rank of T.*

Using the results of Lemma 1, we have an isomorphism

$$\overline{Sp} \simeq \prod \overline{Sp}_\alpha, \tag{22}$$

where $\overline{Sp}_\alpha = Sp(V_\alpha, \varpi_\alpha)$ denotes the group of K_α-linear symplectomorphisms with respect to the form ϖ_α. Moreover, for every $\alpha \in \Xi$ we have $T_\alpha \subset \overline{Sp}_\alpha$. In particular, under the identifications $\overline{Sp}_\alpha \simeq SL_2(K_\alpha)$, there exist the following sequence of inclusions

$$T = \prod T_\alpha \subset \prod SL_2(K_\alpha) = SL_2(K) \subset Sp, \tag{23}$$

and for every $\alpha \in \Xi$ the torus T_α coincides with the K_α-rational points of a maximal torus $\mathbf{T}_\alpha \subset \mathbf{SL}_2$.

3.3 Self-reducibility of the Weil representation

In this subsection we assume that the field k is a finite field of odd characteristic (although, the results continue to hold true also for local fields of characteristic $\neq 2$, i.e., with the appropriate modification, replacing the group

Sp with its double cover $\widetilde{Sp}$). Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation associated with a central character $\psi : Z(J) = Z(H) \rightarrow \mathbb{C}^{\times}$. Recall that $J = Sp \ltimes H$, and τ is obtained as a semi-direct product, $\tau = \rho \ltimes \pi$, of the Weil representation ρ and the Heisenberg representation π. Let $T \subset Sp$ be a maximal torus.

A particular case

For clarity of presentation, assume first that T acts irreducibly on V. Using the results of the previous section, there exists a symplectic module structure $(K, V, \overline{\omega})$ (in this case K/k is a field extension of degree $[K : k] = N$). The group $\overline{Sp} = Sp(V, \overline{\omega})$ is imbedded as a subgroup $\iota_S : \overline{Sp} \hookrightarrow Sp$. Our goal is to describe the restriction

$$(\overline{\rho} = \iota_S^{*}\rho, \overline{Sp}, \mathcal{H}). \tag{24}$$

Define an auxiliary Heisenberg group

$$\overline{H} = V \times K, \tag{25}$$

with the multiplication given by $(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\overline{\omega}(v, v'))$. There exists homomorphism

$$\iota_H : \overline{H} \rightarrow H, \tag{26}$$

given by $(v, z) \mapsto (v, \mathrm{Tr}(z))$. Consider the pullback $(\overline{\pi} = \iota_H^{*}\pi, \overline{H}, \mathcal{H})$. We have

Proposition 3. *The representation $(\overline{\pi} = \iota_H^{*}\pi, \overline{H}, \mathcal{H})$ is the Heisenberg representation associated with the central character $\overline{\psi} = \psi \circ Tr$.*

The group $\overline{Sp}$ acts by automorphisms on the group $\overline{H}$ through its tautological action on the V-coordinate. This action is compatible with the action of Sp on H, i.e., we have $\iota_H(g \cdot h) = \iota_S(g) \cdot \iota_H(h)$ for every $g \in \overline{Sp}$, and $h \in \overline{H}$. The description of the representation $\overline{\rho}$ (24) now follows easily (cf. [9]).

Theorem 9 (Self-reducibility property (particular case)). *The representation $(\overline{\rho}, \overline{Sp}, \mathcal{H})$ is the Weil representation associated with the Heisenberg representation $(\overline{\pi}, \overline{H}, \mathcal{H})$.*

Remark 7. We can summarize the result in a slightly more elegant manner using the Jacobi groups. Let $J = Sp \ltimes H$ and $\overline{J} = \overline{Sp} \ltimes \overline{H}$ be the Jacobi groups associated with the symplectic spaces (V, ω) and $(V, \overline{\omega})$ respectively. We have a homomorphism $\iota : \overline{J} \rightarrow J$, given by $\iota(g, h) = (\iota_S(g), \iota_H(h))$. Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation of J associated with a character ψ of the center $Z(J)$ (note that $Z(J) = Z(H)$), then the pullback $(\iota^{*}\tau, \overline{J}, \mathcal{H})$ is the Heisenberg–Weil representation of $\overline{J}$, associated with the character $\overline{\psi} = \psi \circ \mathrm{Tr}$ of the center $Z(\overline{J})$.

The general case

Here, we drop the assumption that T acts irreducibly on V. Let $(K, V, \overline{\omega})$ be the associated symplectic module structure (19). Using the results of Subsection 3.2, we have decompositions

$$(V, \omega) = \bigoplus_{\alpha \in \Xi} (V_\alpha, \omega_\alpha), \qquad (V, \overline{\omega}) = \bigoplus_{\alpha \in \Xi} (V_\alpha, \overline{\omega}_\alpha), \tag{27}$$

where $\overline{\omega}_\alpha : V_\alpha \times V_\alpha \to K_\alpha$. Let (cf. 25) $\overline{H} = V \times K$, be the Heisenberg group associated with $(V, \overline{\omega})$. There exists (cf. (26) a homomorphism $\iota_H : \overline{H} \to H$. Let us describe the pullback $\overline{\pi} = \iota_H^* \pi$ of the Heisenberg representation. First, we note that the decomposition (27) induces a corresponding decomposition of the Heisenberg group, $\overline{H} = \prod \overline{H}_\alpha$, where $\overline{H}_\alpha$ is the Heisenberg group associated with $(V_\alpha, \overline{\omega}_\alpha)$. We have the following proposition

Proposition 4. *There exists an isomorphism*

$$(\overline{\pi}, \overline{H}, \mathcal{H}) \simeq (\bigotimes \overline{\pi}_\alpha, \prod \overline{H}_\alpha, \bigotimes \mathcal{H}_\alpha),$$

where $(\overline{\pi}_\alpha, \overline{H}_\alpha, \mathcal{H}_\alpha)$ *is the Heisenberg representation of* $\overline{H}_\alpha$ *associated with the central character* $\overline{\psi}_\alpha = \psi \circ Tr_{K_\alpha/k}$.

Let $\iota_S : \overline{Sp} \hookrightarrow Sp$, be the embedding (20). Our next goal is to describe the restriction $\overline{\rho} = \iota_S^* \rho$. Recall that we have a decomposition $\overline{Sp} = \prod \overline{Sp}_\alpha$ (see (22)). In terms of this decomposition we have (cf. [9])

Theorem 10 (Self-reducibility property—general case). *There exists an isomorphism*

$$(\overline{\rho}, \overline{Sp}, \mathcal{H}) \simeq (\bigotimes \overline{\rho}_\alpha, \prod \overline{Sp}_\alpha, \bigotimes \mathcal{H}_\alpha),$$

where $(\overline{\rho}_\alpha, \overline{Sp}_\alpha, \mathcal{H}_\alpha)$ *is the Weil representation associated with the Heisenberg representation* $\overline{\pi}_\alpha$.

Remark 8. As before, we can state an equivalent result using the Jacobi groups $J = Sp \ltimes H$ and $\overline{J} = \overline{Sp} \ltimes \overline{H}$. We have a decomposition $\overline{J} = \prod \overline{J}_\alpha$, where $\overline{J}_\alpha = \overline{Sp}_\alpha \ltimes \overline{H}_\alpha$. Let τ be the Heisenberg–Weil representation of J associated with a character ψ of the center $Z(J)$ (note that $Z(J) = Z(H)$). Then the pullback $\overline{\tau} = \iota^* \tau$ is isomorphic to $\bigotimes \overline{\tau}_\alpha$, where $\overline{\tau}_\alpha$ is the Heisenberg–Weil representation of $\overline{J}_\alpha$, associated with the character $\overline{\psi}_\alpha = \psi \circ \mathrm{Tr}_{K_\alpha/k}$ of the center $Z(\overline{J}_\alpha)$.

3.4 Application to multiplicities

Let us specialize to the case where the filed k is a finite field of odd characteristic. Let $T \subset Sp$ be a maximal torus. The torus T acts, via the Weil representation ρ, on the space $\mathcal{H}$, decomposing it into a direct sum of T-character spaces $\mathcal{H} = \bigoplus_{\chi:T\to\mathbb{C}^\times} \mathcal{H}_\chi$. Consider the problem of determining the multiplicities $m_\chi = \dim(\mathcal{H}_\chi)$. Using Lemma 1, we have (see (23)) a canonical decomposition of T

$$T = \prod T_\alpha, \tag{28}$$

where each of the tori T_α coincides with a maximal torus inside $\overline{Sp} \simeq SL_2(K_\alpha)$, for some field extension $K_\alpha \supset k$. In particular, by (28) we have a decomposition

$$\mathcal{H}_\chi = \bigotimes_{\chi_\alpha:T_\alpha\to\mathbb{C}^\times} \mathcal{H}_{\chi_\alpha}, \tag{29}$$

where $\chi = \prod \chi_\alpha : \prod T_\alpha \to \mathbb{C}^\times$. Hence, by Theorem 10, and the result about the multiplicities in the two-dimensional case (see Theorem (8)), we can compute the integer m_χ as follows. Denote by σ_α the quadratic character of T_α (note that by Theorem (8) the quadratic character σ_α cannot appear in the decomposition (29) if the torus T_α is inert).

Theorem 11. *We have*

$$m_\chi = 2^l,$$

where $l = \#\{\alpha : \chi_\alpha = \sigma_\alpha\}$.

4 Bounds on Higher-Dimensional Exponential Sums

In this section we present an application of the self-reducibility technique to bound higher-dimensional exponential sums attached to tori in $Sp = Sp(V,\omega)$, where (V,ω) is a $2N$-dimensional symplectic vector space over the finite field $\mathbb{F}_p$, $p \neq 2$. These exponential sums originated from the theory of quantum chaos (see Sections 5 and 6). Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation associated with a central character $\psi : Z(J) = Z(H) \to \mathbb{C}^\times$. Recall that $J = Sp \ltimes H$, and τ is obtained as a semi-direct product $\tau = \rho \ltimes \pi$ of the Weil representation ρ and the Heisenberg representation π. Consider a maximal torus $T \subset Sp$. The torus T acts semisimply on $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \bigoplus_{\chi:T\to\mathbb{C}^\times} \mathcal{H}_\chi$. We shall study common eigenstates $\Psi \in \mathcal{H}_\chi$. In particular, we will be interested in estimating matrix coefficients of the form $\langle \Psi|\pi(\xi)\Psi\rangle$ where $\xi \in V$ is not contained in any proper T-invariant subspace. It will be convenient to assume first that the torus T is completely inert (i.e., acts irreducibly on V). In this case one can show (see Theorem 11) that $\dim \mathcal{H}_\chi = 1$ for every χ. Below we sketch a proof of the following estimate.

Theorem 12. *For $\xi \in V$ which is not contained in any proper T-invariant subspace, we have*

$$|\langle \Psi | \pi(\xi) \Psi \rangle| \leq \frac{2 + o(1)}{\sqrt{p}^N}.$$

Let us explain way it is not easy to get such a bound by a direct calculation. Choosing a Schrödinger realization (see Remark 6), we can identify $\mathcal{H} = \mathbb{C}(\mathbb{F}_p^N)$. Under this identification, the matrix coefficient is equivalent to a sum

$$\langle \Psi | \pi(\xi) \Psi \rangle = \sum_{x \in \mathbb{F}_p^N} \Psi(x) e^{\frac{2\pi i}{p} \xi_+ x} \overline{\Psi}(x + \xi_-). \tag{30}$$

In this respect two problems are encountered. First, it is not easy to describe the eigenstates Ψ. Second, the sum (30) is a high-dimensional exponential sum, which is known to be hard to analyze using standard techniques.

Interestingly enough, representation theory suggests a remedy for both problems. Our strategy will be to interpret the matrix coefficient $\langle \Psi | \pi(\xi) \Psi \rangle$ in representation theoretic terms, and then to show, using the self-reducibility technique, that (30) is equivalent to a 1-dimensional sum over $\mathbb{F}_q$, $q = p^N$.

Representation theory and dimensional reduction of (30)

The torus T acts irreducibly on the vector space V. Invoking the result of Section 3.2, there exists a canonical symplectic module structure $(K, V, \overline{\omega})$ associated to T. Recall that in this particular case the algebra K is in fact a field, and $\dim_K V = 2$ (in our case $K = \mathbb{F}_q$, where $q = p^N$). Let $\overline{J} = \overline{Sp} \ltimes \overline{H}$ be the Jacobi group associated to the (two-dimensional) symplectic vector space $(V, \overline{\omega})$ over K. There exists a natural homomorphism $\iota : \overline{J} \to J$. Invoking the results of Section 3.3, the pullback $\overline{\tau} = \iota^* \tau$ is the Heisenberg–Weil representation of $\overline{J}$, i.e., $\overline{\tau} = \overline{\rho} \ltimes \overline{\pi}$.

Let $\Psi \in \mathcal{H}_\chi$. Denote by P_χ the orthogonal projector on the vector space $\mathcal{H}_\chi$. We can write P_χ in terms of the Weil representation $\overline{\rho}$

$$P_\chi = \frac{1}{|T|} \sum_{B \in T} \chi^{-1}(B) \overline{\rho}(B). \tag{31}$$

Since $\dim \mathcal{H}_\chi = 1$ (Theorem 11) we realize that

$$\langle \Psi | \pi(\xi) \Psi \rangle = \mathrm{Tr}(P_\chi \pi(\xi)). \tag{32}$$

Substituting (31) in (32), we can write

$$\langle \Psi | \pi(\xi) \Psi \rangle = \frac{1}{|T|} \sum_{B \in T} \chi^{-1}(B) \mathrm{Tr}(\overline{\rho}(B) \pi(\xi)).$$

Noting that $\mathrm{Tr}(\overline{\rho}(B)\pi(\xi))$ is nothing other than the character $\mathrm{ch}_{\overline{\tau}}(B \cdot \xi)$ of the Heisenberg–Weil representation $\overline{\tau}$. and that $|T| = p^N + 1$, we deduce that it is enough to prove that

$$\left| \sum_{B \in T} \chi^{-1}(B) \mathrm{ch}_{\overline{\tau}}(B \cdot \xi) \right| \leq 2\sqrt{q}, \tag{33}$$

where $q = p^N$. Now, note that the left-hand side of (33) is a one-dimensional exponential sum over $\mathbb{F}_q$, which is defined completely in terms of the two-dimensional Heisenberg–Weil representation $\overline{\tau}$. Estimate (33) is then a particular case of the following theorem, proved in [13].

Theorem 13. *Let (V, ω) be a two-dimensional symplectic vector space over a finite field $k = \mathbb{F}_q$, and $(\tau, J, \mathcal{H})$ be the corresponding Heisenberg–Weil representation. Let $T \subset Sp$ be a maximal torus. We have the following estimate*

$$\left| \sum_{B \in T} \chi(B) ch_{\tau}(B \cdot \xi) \right| \leq 2\sqrt{q}, \tag{34}$$

where χ is a character of T, and $0 \neq \xi \in V$ is not an eigenvector of T.

4.1 General case

In this subsection we state and prove the analogue of Theorem 12, where we drop the assumption of T being completely inert. In what follows, we use the results of Subsections 3.2 and 3.3.

Let $(K, V, \overline{\omega})$ be the symplectic module structure associated with the torus T. The algebra K is no longer a field, but decomposes into a direct sum of fields $K = \bigoplus\limits_{\alpha \in \Xi} K_\alpha$. We have canonical decompositions

$$(V, \omega) = \bigoplus (V_\alpha, \omega_\alpha), \qquad (V, \overline{\omega}) = \bigoplus (V_\alpha, \overline{\omega}_\alpha).$$

Recall that V_α is a two-dimensional vector space over the field K_α. The Jacobi group $\overline{J}$ decomposes into $\overline{J} = \prod \overline{J}_\alpha$, where $\overline{J}_\alpha = \overline{Sp}_\alpha \ltimes \overline{H}_\alpha$ is the Jacobi group associated to $(V_\alpha, \overline{\omega}_\alpha)$. The pullback $(\overline{\tau} = \iota^* \tau, \overline{J}, \mathcal{H})$ decomposes into a tensor product $(\bigotimes \overline{\tau}_\alpha, \prod \overline{J}_\alpha, \bigotimes \mathcal{H}_\alpha)$, where $\overline{\tau}_\alpha$ is the Heisenberg–Weil representation of $\overline{J}_\alpha$. The torus T decomposes into $T = \prod T_\alpha$, where T_α is a maximal torus in $\overline{Sp}_\alpha$. Consequently, the character $\chi : T \to \mathbb{C}^\times$ decomposes into a product $\chi = \Pi \chi_\alpha : \Pi T_\alpha \to \mathbb{C}^\times$, and the space $\mathcal{H}_\chi$ decomposes into a tensor product

$$\mathcal{H}_\chi = \bigotimes \mathcal{H}_{\chi_\alpha}. \tag{35}$$

It follows from the above decomposition that it is enough to estimate matrix coefficients with respect to *pure tensor* eigenstates, i.e., eigenstates Ψ of the form $\Psi = \bigotimes \Psi_\alpha$, where $\Psi_\alpha \in \mathcal{H}_{\chi_\alpha}$. For a vector of the form $\xi = \bigoplus \xi_\alpha$, we have

$$\left\langle \bigotimes \Psi_\alpha | \pi(\xi) \bigotimes \Psi_\alpha \right\rangle = \prod \left\langle \Psi_\alpha | \pi(\xi_\alpha) \Psi_\alpha \right\rangle. \tag{36}$$

Hence, we need to estimate the matrix coefficients $\langle \Psi_\alpha | \pi(\xi_\alpha) \Psi_\alpha \rangle$, but these are defined in terms of the two-dimensional Heisenberg–Weil representation $\overline{\tau}_\alpha$. In addition, we recall the assumption that the vector $\xi \in V$ is not contained in any proper T-invariant subspace. This condition in turn implies that no summand ξ_α is an eigenvector of T_α. Hence, we can use Lemma 13, obtaining

$$|\langle \Psi_\alpha | \pi(\xi_\alpha) \Psi_\alpha \rangle| \leq 2/\sqrt{p}^{[K_\alpha : \mathbb{F}_p]}. \tag{37}$$

Consequently, using (36) and (37) we obtain

$$\left| \left\langle \bigotimes \Psi_\alpha | \pi(\xi) \bigotimes \Psi_\alpha \right\rangle \right| \leq 2^{|\Xi|}/\sqrt{p}^{\sum [K_\alpha : \mathbb{F}_p]} = 2^{|\Xi|}/\sqrt{p}^{[K:\mathbb{F}_p]} = 2^{|\Xi|}/\sqrt{p}^{N}.$$

Recall that the number $r_p = |\Xi|$ is called the symplectic rank of the torus T. The main application of the self-reducibility property, presented in these notes, is summarized in the following theorem.

Theorem 14. *Let (V, ω) be a $2N$-dimensional vector space over the finite field $\mathbb{F}_p$, and $(\tau, J, \mathcal{H})$ the corresponding Heisenberg–Weil representation. Let $\Psi \in \mathcal{H}_\chi$ be a unit χ-eigenstate with respect to a maximal torus $T \subset Sp$. We have the following estimate:*

$$|\langle \Psi | \pi(\xi) \Psi \rangle| \leq \frac{m_\chi \cdot (2 + o(1))^{r_p}}{\sqrt{p}^{N}},$$

where $1 \leq r_p \leq N$ is the symplectic rank of T, $m_\chi = \dim \mathcal{H}_\chi$, and $\xi \in V$ is not contained in any T-invariant subspace.

5 The Hannay–Berry model

We shall proceed to describe the higher-dimensional Hannay–Berry model of quantum mechanics on toral phase spaces. This model plays an important role in the mathematical theory of quantum chaos as it serves as a model where general phenomena, which are otherwise treated only on a heuristic basis, can be rigorously proven.

5.1 The classical phase space

Our classical phase space is the $2N$-dimensional symplectic torus $(\mathbb{T}, \omega)$. We denote by Γ the group of linear symplectic automorphisms of $\mathbb{T}$. Note that $\Gamma \simeq Sp(2N, \mathbb{Z})$. On the torus $\mathbb{T}$ we consider an algebra of complex functions (observables) $\mathcal{A} = \mathcal{F}(\mathbb{T})$. We denote by $\Lambda \simeq \mathbb{Z}^{2N}$ the lattice of characters (exponents) of $\mathbb{T}$. The form ω induces a skew-symmetric form on Λ, which we denote also by ω, and we assume it takes integral values on Λ and is normalized so that $\int_{\mathbb{T}} |\omega|^N = 1$.

5.2 The classical mechanical system

We take our classical mechanical system to be of a very simple nature. Let $A \in \Gamma$ be a generic element (see Definition 1), i.e., A is regular and admits no invariant co-isotropic sub-tori. The last condition can be equivalently restated in dual terms, namely, requiring that A admits no invariant isotropic subvectorspaces in $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The element A generates, via its action as an automorphism $A : \mathbb{T} \longrightarrow \mathbb{T}$, a discrete time dynamical system.

5.3 Quantization

Before we employ the formal model, it is worthwhile to discuss the general phenomenological principles of quantization which are common to all models. Principally, quantization is a protocol by which one associates a Hilbert space $\mathcal{H}$ to the classical phase space, which in our case is the torus $\mathbb{T}$; In addition, the protocol gives a rule

$$f \rightsquigarrow Op(f) : \mathcal{H} \to \mathcal{H},$$

by which one associates an operator on the Hilbert space to every classical observable, i.e., a function $f \in \mathcal{F}(\mathbb{T})$. This rule should send a real function into a self-adjoint operator. In addition, in the presence of classical symmetries which in our case are given by the group Γ, the Hilbert space $\mathcal{H}$ should support a (projective unitary) representation $\Gamma \to PGL(\mathcal{H})$,

$$\gamma \mapsto U(\gamma) : \mathcal{H} \to \mathcal{H},$$

which is compatible with the quantization rule $Op(\cdot)$.

More precisely, quantization is not a single protocol, but a one-parameter family of protocols, parameterized by a parameter $\hbar$ called the Planck constant. Accepting these general principles, one searches for a formal model by which to quantize. In this work we employ a model called the *non-commutative torus* model.

5.4 The non-commutative torus model

Denote by $\mathcal{A}$ the algebra of trigonometric polynomials on $\mathbb{T}$, i.e., $\mathcal{A}$ consists of functions f which are a finite linear combinations of characters. We shall construct a one-parametric deformation of $\mathcal{A}$ called the non-commutative torus [6, 29].

Let $\hbar = 1/p$, where p is an odd prime number, and consider the additive character $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^\times$, $\psi(t) = e^{\frac{2\pi i t}{p}}$. We give here the following presentation of the algebra $\mathcal{A}_\hbar$. Let $\mathcal{A}_\hbar$ be the free non-commutative $\mathbb{C}$-algebra generated by the symbols $s(\xi)$, $\xi \in \Lambda$, and the relations

$$s(\xi)s(\eta) = \psi(\tfrac{1}{2}\omega(\xi,\eta))s(\xi+\eta). \tag{38}$$

Here we consider ω as a map $\omega : \Lambda \times \Lambda \longrightarrow \mathbb{F}_p$.

Note that $\mathcal{A}_\hbar$ satisfies the following properties:

- As a vector space $\mathcal{A}_\hbar$ is equipped with a natural basis $s(\xi)$, $\xi \in \Lambda$. Hence we can identify the vector space $\mathcal{A}_\hbar$ with the vector space $\mathcal{A}$ for each value of $\hbar$,

$$\mathcal{A}_\hbar \simeq \mathcal{A}. \tag{39}$$

- Substituting $\hbar = 0$ we have $\mathcal{A}_0 = \mathcal{A}$. Hence, we see that indeed $\mathcal{A}_\hbar$ is a deformation of the algebra of (algebraic) functions on $\mathbb{T}$.
- The group Γ acts by automorphisms on the algebra $\mathcal{A}_\hbar$, via $\gamma \cdot s(\xi) = s(\gamma\xi)$, where $\gamma \in \Gamma$ and $\xi \in \Lambda$. This action induces an action of Γ on the category of representations of $\mathcal{A}_\hbar$, taking a representation π and sending it to the representation π^γ, where $\pi^\gamma(f) = \pi(\gamma f)$, $f \in \mathcal{A}_\hbar$.

Using the identification (39) , we can describe a choice for the quantization of the functions. We just need to pick a representation of the quantum algebra $\mathcal{A}_\hbar$. But what representation to pick? It turns out that, we have a canonical choice. All the irreducible algebraic representations of $\mathcal{A}_\hbar$ are classified [12] and each of them is of dimension p^N. We have

Theorem 15 (Invariant representation [12]). *Let $\hbar = 1/p$ where p is a prime number. There exists a unique (up to isomorphism) irreducible representation $\pi : \mathcal{A}_\hbar \to End(\mathcal{H}_\hbar)$ which is fixed by the action of Γ. Namely, π^γ is isomorphic to π for every $\gamma \in \Gamma$.*

Let $(\pi, \mathcal{A}_\hbar, \mathcal{H})$ be a representative of the special representation defined in Theorem 15. For every element $\gamma \in \Gamma$ we have an isomorphism $\widetilde{\rho}(\gamma) : \mathcal{H} \to \mathcal{H}$ intertwining the representations π and π^γ , namely, it satisfies $\widetilde{\rho}(\gamma)\pi(f)\widetilde{\rho}(\gamma)^{-1} = \pi(\gamma f)$, for every $f \in \mathcal{A}_\hbar$ and $\gamma \in \Gamma$. The isomorphism $\widetilde{\rho}(\gamma)$ is not unique but unique up to a scalar (this is a consequence of Schur's lemma and the fact that π and π^γ are irreducible representations). It is easy to realize that the collection $\{\widetilde{\rho}(\gamma)\}$ constitutes a projective representation $\widetilde{\rho} : \Gamma \to PGL(\mathcal{H})$. Let $\hbar = 1/p$ where p is an odd prime $\neq 3$. We have the following linearization theorem (cf. [11, 13])

Theorem 16 (Linearization). *The projective representation $\widetilde{\rho}$ can be linearized uniquely to an honest representation $\rho : \Gamma \to GL(\mathcal{H})$ that factors through the finite quotient group $Sp \simeq Sp(2N, \mathbb{F}_p)$.*

Remark 9. The representation $\rho : Sp \to GL(\mathcal{H})$ is the celebrated Weil representation, here obtained via quantization of the torus.

5.5 The quantum dynamical system

Recall that we started with a classical dynamic on $\mathbb{T}$, generated by a generic (i.e., regular with no non-trivial invariant co-isotropic sub-tori) element $A \in \Gamma$. Using the Weil representation, we can associate to A the unitary operator

$\rho(A) : \mathcal{H} \to \mathcal{H}$, which constitutes the generator of discrete time quantum dynamics. We would like to study the $\rho(A)$-eigenstates

$$\rho(A)\Psi = \lambda\Psi,$$

which satisfy additional symmetries. This we do in the next section.

6 The Hecke quantum unique ergodicity theorem

It turns out that the operator $\rho(A)$ has degeneracies namely, its eigenspaces might be extremely large. This is manifested in the existence of a group of hidden symmetries commuting with $\rho(A)$ (note that classically the group of linear symplectomorphisms of $\mathbb{T}$ that commute with A, i.e., $\mathbf{T}_A(\mathbb{Z})$, does not contribute much to the harmonic analysis of $\rho(A)$). These symmetries can be computed. Indeed, let $T_A = Z(A, Sp)$, be the centralizer of the element A in the group Sp. Clearly T_A contains the cyclic group $\langle A \rangle$ generated by the element A, but it often happens that T_A contains additional elements. The assumption that A is regular (i.e., has distinct eigenvalues) implies that for sufficiently large p the group T_A consists of the $\mathbb{F}_p$-rational points of a maximal torus $\mathbf{T}_A \subset \mathbf{Sp}$, i.e., $T_A = \mathbf{T}_A(\mathbb{F}_p)$ (more precisely, p large enough so that it does not divides the discriminant of A). The group T_A is called the *Hecke* torus. It acts semisimply on $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \bigoplus\limits_{\chi : T_A \to \mathbb{C}^\times} \mathcal{H}_\chi$. We shall study common eigenstates $\Psi \in \mathcal{H}_\chi$, which we call *Hecke eigenstates* and will be assumed to be normalized so that $\|\Psi\|_{\mathcal{H}} = 1$. In particular, we will be interested in estimating matrix coefficients of the form $\langle \Psi | \pi(f)\Psi \rangle$, where $f \in \mathcal{A}$ is a classical observable on the torus $\mathbb{T}$ (see Subsection 5.4). We will call these matrix coefficients *Hecke–Wigner distributions*. It will be convenient for us to start with the following case.

6.1 The strongly generic case

Let us assume first that the automorphism A acts on $\mathbb{T}$ with no invariant sub-tori. In dual terms, this means that the element A acts irreducibly on the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

We denote by r_p the symplectic rank of T_A, i.e., $r_p = |\Xi|$ where $\Xi = \Xi(T_A)$ is the symplectic type of T_A (see Definition 3). By definition we have $1 \leq r_p \leq N$ (for example, we get the two extreme cases: $d_p = 1$ when the torus T_A acts irreducibly on $V \simeq \mathbb{F}_p^{2N}$, and $d_p = N$ when T_A splits). We have

Theorem 17. *Consider a non-trivial exponent $0 \neq \xi \in \Lambda$ and a sufficiently large prime number p. Then for every normalized Hecke eigenstate $\Psi \in \mathcal{H}_\chi$ the following bound holds:*

$$|\langle \Psi|\pi(\xi)\Psi\rangle| \leq \frac{m_\chi \cdot 2^{r_p}}{\sqrt{p}^N}, \tag{40}$$

where $m_\chi = \dim(\mathcal{H}_\chi)$.

The lattice Λ constitutes a basis for $\mathcal{A}$, hence, using the bound (40) we obtain

Corollary 5 (Hecke quantum unique ergodicity—strongly generic case). *Consider an observable* $f \in \mathcal{A}$ *and a sufficiently large prime number* p. *For every normalized Hecke eigenstate* Ψ *we have*

$$\left| \langle \Psi|\pi(f)\Psi\rangle - \int_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^N},$$

where $\mu = |\omega|^N$ *is the corresponding volume form and* C_f *is an explicit computable constant which depends only on the function* f.

Remark 10. In Subsection 6.2 we will elaborate on the distribution of the symplectic rank r_p (40) and in Subsection 6.3 the more general statements where $A \in \Gamma$ is any generic element (see Definition 1) will be stated and proved.

Proof of Theorem 17

The proof is by reduction to the bound on the Hecke–Wigner distributions obtained in Section 4, namely reduction to Theorem 14. Our first goal is to interpret the Hecke–Wigner distribution $\langle \Psi|\pi(\xi)\Psi\rangle$ in terms of the Heisenberg–Weil representation.

Step 1. *Replacing the non-commutative torus by the finite Heisenberg group.* Note that the Hilbert space $\mathcal{H}$ is a representation space of both the algebra $\mathcal{A}_\hbar$ and the group Sp. We will show next that the representation $(\pi, \mathcal{A}_\hbar, \mathcal{H})$ is *equivalent* to the Heisenberg representation of some finite Heisenberg group. The representation π is determined by its restriction to the lattice Λ. However, the restriction

$$\pi_{|\Lambda} : \Lambda \to GL(\mathcal{H}),$$

is not multiplicative and in fact constitutes (see Formula (38)) a projective representation of the lattice given by

$$\pi(\xi)\pi(\eta) = \psi(\tfrac{1}{2}\omega(\xi,\eta))\pi(\xi+\eta). \tag{41}$$

It is evident from (41) that the map $\pi_{|\Lambda}$ factors through the quotient $\mathbb{F}_p$-vector space V

$$\Lambda \to V = \Lambda/p\Lambda \to GL(\mathcal{H}). \tag{42}$$

The vector space V is equipped with a symplectic structure ω obtained via specialization of the corresponding form on Λ. Let $H = H(V,\omega)$ be the Heisenberg group associated with (V,ω). Recall that as a set $H = V \times \mathbb{F}_p$ and the multiplication is given by

$$(v,z)\cdot(v',z') = (v+v', z+z'+\tfrac{1}{2}\omega(v,v')). \tag{43}$$

From formula (41), the factorization (42), and the multiplication rule (43) we learn that the map $\pi : V \to GL(\mathcal{H})$, given by (42), lifts to an honest representation of the Heisenberg group $\pi : H \to GL(\mathcal{H})$. Finally, the pair (ρ,π), where ρ is the Weil representation obtained using quantization of the torus (see Theorem 16) glues into a single representation $\tau = \rho \ltimes \pi$ of the Jacobi group $J = Sp \ltimes H$, which is of course nothing other than the Heisenberg–Weil representation

$$\tau : J \to GL(\mathcal{H}). \tag{44}$$

Having the Heisenberg–Weil representation at our disposal we proceed to

Step 2.*Reformulation.* Let $\mathbf{V}$ and $\mathbf{T}_A$ be the algebraic group scheme defined over $\mathbb{Z}$ so that $\Lambda = \mathbf{V}(\mathbb{Z})$ and for every prime p we have $V = \mathbf{V}(\mathbb{F}_\mathbf{p})$ and $T_A = \mathbf{T}_A(\mathbb{F}_p)$. In this setting for every prime number p we can consider the lattice element $\xi \in \Lambda$ as a vector in the $\mathbb{F}_p$-vector space V.

Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation (44) and consider a normalized Hecke eigenstate $\Psi \in \mathcal{H}_\chi$. We need to verify that for a sufficiently large prime number p we have

$$|\langle \Psi|\pi(\xi)\Psi\rangle| \leq \frac{m_\chi \cdot 2^{r_p}}{\sqrt{p}^N}, \tag{45}$$

where m_χ denotes the multiplicity $m_\chi = \dim \mathcal{H}_\chi$ and r_p is the symplectic rank of T_A. This verification is what we do next.

Step 3. *Verification.* We need to show that we meet the conditions of Theorem 14. What is left to check is that for sufficiently large prime number p the vector $\xi \in V$ is not contained in any T_A-invariant subspace of V. Let us denote by O_ξ the orbit $O_\xi = T_A \cdot \xi$. We need to show that for sufficiently large p we have

$$\mathrm{Span}_{\mathbb{F}_p}\{O_\xi\} = V. \tag{46}$$

The condition (46) is satisfied since it holds globally. In more details, our assumption on A guarantees that it holds for the corresponding objects over the field of rational numbers $\mathbb{Q}$, i.e., $\mathrm{Span}_{\mathbb{Q}}\{\mathbf{T}_A(\mathbb{Q})\cdot\xi\} = \mathbf{V}(\mathbb{Q})$. Hence (46) holds for a sufficiently large prime number p.

6.2 The distribution of the symplectic rank

We would like to compute the asymptotic distribution of the symplectic rank r_p (45) in the set $\{1,\ldots,N\}$, i.e.,

$$\delta(r) = \lim_{x\to\infty} \frac{\#\{r_p = r\ ;\ p \le x\}}{\pi(x)}, \tag{47}$$

where $\pi(x)$ denotes the number of prime numbers up to x.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$, and denote by G the Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Consider the vector space $\mathbf{V} = \mathbf{V}(\overline{\mathbb{Q}})$. By extension of scalars the symplectic form ω on $\mathbf{V}(\mathbb{Q})$ induces a $\overline{\mathbb{Q}}$-linear symplectic form on $\mathbf{V}$, which we will also denote by ω. Let $\mathbf{T}$ denote the algebraic torus $\mathbf{T} = \mathbf{T}_A(\overline{\mathbb{Q}})$. The action of $\mathbf{T}$ on $\mathbf{V}$ is completely reducible, decomposing it into one-dimensional character spaces $\mathbf{V} = \bigoplus_{\chi\in\mathfrak{X}} \mathbf{V}_\chi$.

Let Θ be the restriction of the symplectic transpose $(\cdot)^t : \mathrm{End}(\mathbf{V}) \to \mathrm{End}(\mathbf{V})$ to $\mathbf{T}$. The involution Θ acts on the set of characters $\mathfrak{X}$ by $\chi \mapsto \Theta(\chi) = \chi^{-1}$ and this action is compatible with the action of the Galois group G on $\mathfrak{X}$ by conjugation $\chi \mapsto g\chi g^{-1}$, where $\chi \in \mathfrak{X}$ and $g \in G$. This means (recall that A is strongly generic) that we have a transitive action of G on the set $\mathfrak{X}/\Theta$. Consider the kernel $\mathrm{K} = \ker(G \to \mathrm{Aut}(\mathfrak{X}/\Theta))$, and the corresponding finite Galois group $Q = G/\mathrm{K}$. Considering Q as a subgroup of $\mathrm{Aut}(\mathfrak{X}/\Theta)$ we define the cycle number $c(C)$ of a conjugacy class $C \subset Q$ to be the number of irreducible cycles that compose a representative of C. By a direct application of the Chebotarev theorem [5] we get

Proposition 5 (Chebotarev's theorem). *The distribution δ (47) obeys*

$$\delta(r) = \frac{|C_r|}{|Q|},$$

where $C_r = \bigcup_{\substack{C\subset Q\\ c(C)=r}} C$.

6.3 The general generic case

Let us now treat the more general case where the automorphism A acts on $\mathbb{T}$ in a generic way (Definition 1). In dual terms, this means that the torus $\mathbf{T}(\mathbb{Q}) = \mathbf{T}_A(\mathbb{Q})$ acts on the symplectic vector space $\mathbf{V}(\mathbb{Q}) = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ decomposing it into an orthogonal symplectic direct sum

$$(\mathbf{V}(\mathbb{Q}), \omega) = \bigoplus_{\alpha\in\Xi} (\mathbf{V}_\alpha(\mathbb{Q}), \omega_\alpha), \tag{48}$$

with an irreducible action of $\mathbf{T}(\mathbb{Q})$ on each of the spaces $\mathbf{V}_\alpha(\mathbb{Q})$. For an exponent $\xi \in \Lambda$ define its support with respect to the decomposition (48) by $S_\xi = \mathrm{Supp}(\xi) = \{\alpha;\ P_\alpha \xi \neq 0\}$, where $P_\alpha : \mathbf{V}(\mathbb{Q}) \to \mathbf{V}(\mathbb{Q})$ is the projector onto the space $\mathbf{V}_\alpha(\mathbb{Q})$ and denote by d_ξ the dimension $d_\xi = \sum_{\alpha\in S_\xi} \dim \mathbf{V}_\alpha(\mathbb{Q})$.

The decomposition (48) induces a decomposition of the torus $\mathbf{T}(\mathbb{Q})$ into a product of completely inert tori

$$\mathbf{T}(\mathbb{Q}) = \prod_{\alpha \in \Xi} \mathbf{T}_\alpha(\mathbb{Q}). \tag{49}$$

Consider now a sufficiently large prime number p and specialize all the objects involved to the finite filed $\mathbb{F}_p$. The Hecke torus $T = \mathbf{T}(\mathbb{F}_p)$ acts on the quantum Hilbert space $\mathcal{H}$ decomposing it into an orthogonal direct sum $\mathcal{H} = \bigoplus\limits_{\chi : T \to \mathbb{C}^\times} \mathcal{H}_\chi$. The decomposition (49) induces decompositions on the level of groups of points $T = \prod\limits_{\alpha \in \Xi} T_\alpha$, where $T_\alpha = \mathbf{T}_\alpha(\mathbb{F}_p)$, on the level of characters $\chi = \mathop{\Pi}\limits_{\alpha} \chi : \prod\limits_{\alpha} T_\alpha \to \mathbb{C}^\times$, and on the level of character spaces $\mathcal{H}_\chi = \bigotimes\limits_{\alpha} \mathcal{H}_{\chi_\alpha}$.

For each torus T_α we denote by $r_{p,\alpha} = r_p(T_\alpha)$ its symplectic rank (see Definition 3) and we consider the integer $|S_\xi| \leq r_{p,\xi} \leq d_\xi$ given by $r_{p,\xi} = \mathop{\Pi}\limits_{\alpha \in S_\xi} r_{p,\alpha}$.

Let us denote by m_{χ_ξ} the dimension $m_{\chi_\xi} = \sum_{\alpha \in S_\xi} \dim \mathcal{H}_{\chi_\alpha}$. Finally, we can state the theorem for the generic case. We have

Theorem 18 (Hecke quantum unique ergodicity—generic case). *Consider a non-trivial exponent $0 \neq \xi \in \Lambda$ and a sufficiently large prime number p. Then for every normalized Hecke eigenstate $\Psi \in \mathcal{H}_\chi$ the following bound holds:*

$$|\langle \Psi | \pi(\xi) \Psi \rangle| \leq \frac{m_{\chi_\xi} \cdot 2^{r_{p,\xi}}}{\sqrt{p}^{d_\xi}}. \tag{50}$$

Considering the decomposition (48) we denote by d the dimension $d = \min_\alpha \mathbf{V}_\alpha(\mathbb{Q})$. Since the lattice Λ constitutes a basis for the algebra $\mathcal{A}$ of observables on $\mathbb{T}$, then using the bound (50) we obtain

Corollary 6. *Consider an observable $f \in \mathcal{A}$ and a sufficiently large prime number p. Then for every normalized Hecke eigenstate Ψ we have*

$$\left| \langle \Psi | \pi(f) \Psi \rangle - \int\limits_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^d},$$

where $\mu = |\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

The proof of Theorem 18 is a straightforward application of Theorem 17. Indeed, considering the decomposition (48) of the torus $\mathbf{T}(\mathbb{Q})$ to a product of completely inert tori $\mathbf{T}_\alpha(\mathbb{Q})$, we may apply the theory developed for the strongly generic case in Subsection (6.1) to each of the tori $\mathbf{T}_\alpha(\mathbb{Q})$ to deduce Theorem 18.

Remark 11. As explained in Subsection (6.2) the distribution of the symplectic rank $r_{p,\xi}$ is determined by the Chebotarev theorem applied to (now a product of) suitable finite Galois groups Q_α attached to the tori $\mathbf{T}_\alpha$, $\alpha \in S_\xi$ (49).

Remark 12. The corresponding quantum unique ergodicity theorem for statistical states of generic automorphism A of $\mathbb{T}$ (see Theorem 4) follows directly from Theorem 18.

References

1. Adams J. and Moy A., Unipotent representations and reductive dual pairs over finite fields. *Trans. Amer. Math. Soc. 340, no. 1, (1993), 309–321*
2. Berry M.V., Regular and irregular semiclassical wavefunctions. *J. Phys. A 10, no. 12* (1977), 2083–2091.
3. Berry M.V., Semiclassical Mechanics of regular and irregular motion. *Les Houches Lecture Series Session XXXVI, eds. G. Iooss, R.H.G. Helleman and R. Stora, North Holland, Amsterdam* (1983), 171–271.
4. Bouzouina A. and De Biévre S., Equipartition of the eigenstates of quantized ergodic maps on the torus. *Commun. Math. Phys. 178* (1996), 83–105.
5. Chebotarev N.G., Determination of the density of the set of primes corresponding to a given class of permutations, *Izv. Akad. Nauk., 17* (1923), 205–230.
6. Connes A., Algebres et geometrie differentielle. *C. R. Acad. Sci. Paris Ser. A-B 290 (1980), no. 13, A599–A604.*
7. Deligne P., Metaplectique. *A letter to Kazhdan* (1982).
8. Faure F., Nonnenmacher S. and De Bievre S. Scarred eigenstates for quantum cat maps of minimal periods. *Commun. Math. Phys. 239* (2003), 449–492.
9. Gérardin P., Weil representations associated to finite fields. *J. Algebra 46*, no. 1, (1977), 54–101.
10. Gurevich S., Weil representation, Deligne's sheaf, and proof of the Kurlberg-Rudnick conjecture. *Ph.D. Thesis* (submitted: March 2005)
11. Gurevich S. and Hadani R., On Hannay-Berry equivariant quantization of the Torus. *arXiv:math-ph/0312039* (2001).
12. Gurevich S. and Hadani R., On Hannay-Berry equivariant quantization of higher-dimensional tori. *arXiv:math-ph/0403036* (2002).
13. Gurevich S. and Hadani R., Proof of the Kurlrberg-Rudnick rate conjecture. Comptes Rendus Mathematique, Volume 342, Issue 1, (2006), Pages 69–72.
14. Gurevich S. and Hadani R., Hecke quantum unique ergodicity for the higher-dimensional Hannay-Berry models. *arXiv:math-ph/0403036* (2004).
15. Gurevich S. and Hadani R., Heisenberg realizations, eigenfunctions and proof of the Kurlberg-Rudnick Supremum Conjecture. *arXiv:math-ph/0511036* (2005).
16. Gurevich S. and Hadani R., The geometric Weil representation. *Selecta Mathematica, New Series, Birkhäuser Basel* (Accepted: Dec. 2006).
17. Gurevich S. and Hadani R., Quantization of symplectic vector spaces over finite fields. *arXiv:0705.4556* (2005).
18. Hannay J.H. and Berry M.V., Quantization of linear maps on the torus—Fresnel diffraction by a periodic grating. *Physica D1* (1980), 267–291.
19. Heath-Brown R., Private communication. Montreal, July 2005.
20. Heller J.E., Bound-state eigenstates of classically chaotic Hamiltonian systems: scars of periodic orbits. *Phys. Rev. Lett. 53* (1984), 1515–1518.
21. Howe R., Invariant theory and duality for classical groups over finite fields with applications to their singular representation theory, preprint, Yale University.

22. Howe R., On the character of Weil's representation. *Trans. Amer. Math. Soc. 177* (1973), 287–298.
23. Kelmer D., Arithmetic Quantum Unique Ergodicity for Symplectic Linear Maps of the Multidimensional Torus. *math-ph/0510079.*
24. Kaplan L. and Heller E.J. Measuring scars of periodic orbits. *Phys. Rev.* E 59 (1999).
25. Kurlberg P. and Rudnick Z., Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. Jour. 103* (2000), 47–78.
26. Marklof J., Arithmetic quantum chaos. Encyclopedia of Mathematical Physics, eds. J.-P. Francoise, G.L. Naber and Tsou S.T. Oxford: Elsevier, Volume 1 (2006), 212–220.
27. Rudnick Z., The quantized cat map and quantum ergodicity. *Lecture at the MSRI conference—Random Matrices and their Applications, Berkeley, June 7–11, 1999.*
28. Rudnick Z., On quantum unique ergodicity for linear maps of the torus. *European Congress of Mathematics, Vol. II (Barcelona, 2000), Progr. Math., 202, Birkhäuser, Basel.* (2001), 429–437.
29. Rieffel M.A., Non-commutative tori—a case study of non-commutative differentiable manifolds, *Contemporary Math. 105* (1990), 191–211.
30. Sarnak P., Arithmetic quantum chaos. *The Schur lectures, 1992, Tel Aviv. Israel Math. Conf. Proc., 8, Bar-Ilan Univ., Ramat Gan* (1995) 183–236.
31. Shnirelman A. I., Ergodic properties of eigenstates. *Uspehi Mat. Nauk 29, no. 6(180)* (1974), 181–182.
32. Srinivasan B., Weil representations of finite classical groups. *Invent. Math. 51, no. 2, 143–153* (1979).
33. von Neumann J., Mathematical foundations of quantum mechanics. *Princeton University Press* (1955).
34. Weil A., Sur certains groupes d'operateurs unitaires. *Acta Math. 111* (1964) 143–211.
35. Weyl H., Gruppentheorie und Quantenmechanik. *1st edition Hirzel, Leipzig* (1928).
36. Weyl H., The Classical Groups. Their Invariants and Representations. *Princeton University Press, Princeton, N.J.* (1939).