# Introduction to Representation Theory <br> HW3 - Fall 2016 <br> For Thursday Dec. 1 Meeting 

1. Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$.
(a) Recall the definition of the trace morphism $\operatorname{Tr}: \operatorname{End}(V) \rightarrow \mathbb{F}$.
(b) Describe the canonical isomorphism $*: \operatorname{End}(V) \rightarrow \operatorname{End}\left(V^{*}\right)$. Show that $\operatorname{Tr}(T)=$ $\operatorname{Tr}\left(T^{*}\right)$, where $T^{*}$ is defined using the morphism $*$.
(c) Let $(\pi, V),\left(\pi^{*}, V^{*}\right)$ be a representation, and its dual, of a finite group $G$. Show that

$$
\chi_{\pi^{*}}(g)=\chi_{\pi}\left(g^{-1}\right), \quad g \in G .
$$

2. Let $V$ be a finite dimensional vector space over $\mathbb{C}$.
(a) Define the complex conjugate vector space $\bar{V}$ with the same additive structure, but scalar product $\alpha \cdot v:=\bar{\alpha} v$.
(b) Show that $\bar{V}$ and $V^{*}$ are canonically isomorphic iff $V$ is equipped with an inner product $\langle$,$\rangle .$
(c) Let $(\pi, V)$ be a representation of a finite group $G$. Define the representation $\bar{\pi}$ of $G$ on $\bar{V}$ and show that $\pi^{*} \simeq \bar{\pi}$.
3. Let $\pi$ be a representation of a finite group $G$.
(a) Show that there exist a unique idempotent $e_{\pi} \in \mathcal{A}(G)$, such that

$$
\rho\left(e_{\pi}\right)=\left\{\begin{array}{c}
I d_{\rho}, \text { if } \rho \simeq \pi ; \\
0, \text { other } .
\end{array}\right.
$$

(b) Compute the formula of $e_{\pi}$.
4. Let $A$ be a finite Abelian group.
(a) Consider the inner product on $L(A)$

$$
\langle f, h\rangle=\frac{1}{\# A} \sum_{a \in A} f(a) \overline{h(a)} .
$$

Verify the Plancherel formula $\langle F(f), F(f)\rangle^{2}=\# A\langle f, f\rangle$.
(b) Recall the definition of the Pontryagin dual group $\widehat{A}$.
(c) Consider the Fourier transform $F: L(A) \rightarrow L(\widehat{A})$ given by

$$
F(f)[\psi]=\sum_{a \in A} f(a) \psi(a) .
$$

Verify the Fourier inversion formula

$$
f(a)=\sum_{\psi \in \widehat{A}} F(f)[\psi] \psi^{-1}(a) .
$$

5. Let $\mathbb{F}=\mathbb{F}_{q}$. Let $V$ be a finite dimensional vector space over $\mathbb{F}$. Describe the dual group $\widehat{V}$ explicitly as follows. Fix a non-trivial additive character $\psi_{0} \in \mathbb{F} \rightarrow \mathbb{C}^{*}$. Show that the homomorphism

$$
\iota_{0}: V^{*} \rightarrow \widehat{V}, \quad \iota_{0}(\xi)[v]=\psi_{0}(\xi(v))
$$

is an isomorphism.
6. Consider the natural representation $\sigma$ of group $\mathbb{F}^{*}=\mathbb{F}_{q}^{*}$ on the space $\mathcal{H}=L(\mathbb{F})$ of complex valued functions on $\mathbb{F}=\mathbb{F}_{q}$.
(a) Describe explicitly the decomposition of this space into irreducible components. Namely for any multiplicative character $\chi$ of $\mathbb{F}^{*}$ describe the subspace $\mathcal{H}_{\chi} \subset \mathcal{H}$ of functions which transform according to $\chi$.
(b) Show that the Fourier transform $F$ takes $\mathcal{H}_{\chi}$ into $\mathcal{H}_{\chi^{-1}}$.
(c) Using the Plancherel formula prove the following Gauss's identity. Fix a nontrivial additive character $\psi$ of $\mathbb{F}$ and a non-trivial multiplicative character $\chi$ of $\mathbb{F}^{*}$. Then the numerical value of

$$
G=\sum_{a \in \mathbb{F}^{*}} \psi(a) \chi(a),
$$

satisfies $|G|=\sqrt{q}$.

## Good Luck!

