Introduction to Representation Theory
HW2 - Fall 2016
For Thursday October 20 Meeting

1. *Finite Symplectic Spaces.* A symplectic vector space over a field k is a pair $(V, \langle , \rangle)$, where V is a finite dimensional vector space over k, and $\langle , \rangle$ is a non-degenerate bilinear form $\langle , \rangle$ on V, which is alternating, i.e., $\langle u, v \rangle = -\langle v, u \rangle$ for every $u, v \in V$. Such a form is also called symplectic form.

   (a) Define the category of symplectic vector spaces over k and show that every morphism there is injection.

   (b) Show that if V is symplectic then $\dim(V) = 2n$, for some n.

   (c) Show that any two symplectic vector spaces of the same dimension are symplectomorphic, i.e., isomorphic by a symplectic morphism.

   (d) Show that if $I \subset V$ is a subspace on which the symplectic form vanishes then $\dim(I) \leq n$. Such subspace are called isotropic. A maximal isotropic subspace $L \subset V$ is called Lagrangian. For such L show that $\dim(L) = n$.

   (e) Denote by $Lag(V)$ the space of Lagrangians in V. Try to compute $\#Lag(V)$ in case $k = \mathbb{F}_q$.

2. *Heisenberg Group.* Let $(V, \langle , \rangle)$ be a symplectic space over a field k. Let us assume that $char(k) \neq 2$. Consider the set $H = V \times k$ equipped with the product

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2} \langle v, v' \rangle).$$

   (a) Show that H is a group. It is called the Heisenberg group associated with $(V, \langle , \rangle)$.

   (b) Compute the center $Z = Z(H)$ of H, and the commutator subgroup $[H, H]$.

   (c) Show that there exists a natural bijection between maximal commutative subgroups of H and Lagrangians in V.

   (d) Show that the symplectic group $Sp(V)$ acts naturally on H by group automorphisms.

   (e) Describe the set of conjugacy classes of H.

3. *Representations of the finite Heisenberg Group.* Let H be the Heisenberg group associated with a $2n$-dimensional symplectic vector space $(V, \langle , \rangle)$ over $k = \mathbb{F}_q$.

   (a) Compute the number of conjugacy classes of H in the case $k = \mathbb{F}_q$ a finite field, where q is odd.

   (b) Construct q^{2n} one dimensional irreducible representations of H.

   (c) Proof the following **Theorem (Stone–von Neumann).** Irreducible representations of H which are non-trivial on $Z = Z(H)$ and agree there are isomorphic. Moreover, for every nontrivial character $\psi : Z \to \mathbb{C}^*$ there exists a unique (up to isomorphism) irreducible representation $(\pi_\psi, \mathcal{H}_\psi)$ of H with $\pi_\psi(z)) = \psi(z) Id_{\mathcal{H}_\psi}$.

(d) Deduce that a representation $\mathcal{H}$ of H with $\dim(\mathcal{H}) > 1$ is irreducible iff it is of dimension q^n.

(e) Compute the character χ_{π_ψ} for π_ψ as in the theorem.

(f) Let L be a Lagrangian in V. For a fixed $1 \neq \psi : Z \to \mathbb{C}^*$ consider the space

$$\mathcal{H}_{L,\psi} = \{f : H \to \mathbb{C};\ f(l \cdot z \cdot h) = \psi(z)f(h) \text{ for every } l \in L, z \in Z, h \in H\}.$$

Note that the group H acts naturally on $\mathcal{H}_{L,\psi}$ by $[\pi_{L,\psi}(h')f](h) = f(hh')$. Show that $(\pi_{L,\psi}, \mathcal{H}_{L,\psi})$ is irreducible.

(g) Construct a model for each member of $Irr(H)$.

Good Luck!