Introduction to Representation Theory
HW1 - Fall 2016
For Thursdays Sep. 22&29 Meetings

Unless stated otherwise, all groups and sets are finite, and all vector spaces are complex and finite dimensional.

Definition For a representation (ρ, G, V), we denote by $\chi_\rho : G \to \mathbb{C}$ the function $\chi_\rho(g) = Tr(\rho(g))$, and call it the character of ρ.

1. *Permutation Representations.* For a group G and a G-set X, we denote by $(\pi_X, L(X))$ the associated permutation representation, given by

$$[\pi_X(g)f](x) = f(g^{-1}x),$$

where $f \in L(X)$ - the space of complex valued functions on X.

   (a) Verify the formula

$$\chi_{\pi_X}(g) = \#X^g = \#\{x \in X;\ gx = x\}.$$

   Definition The intertwining number of two representation π and ρ of G, denoted $\langle \pi, \rho \rangle$, is the integer

$$\langle \pi, \rho \rangle = \dim \ Hom(\pi, \rho).$$

   (b) Consider two permutation representations $(\pi_{X_j}, L(X_j))$, $j = 1, 2$, of G. Verify the intertwining number formula

$$\langle \pi_{X_1}, \pi_{X_2} \rangle = \# (X_1 \times X_2) / G,$$

   i.e., the number of G-orbits in $X_1 \times X_2$ with respect to the diagonal action.

   (c) For $G = S_n$ and $X = \{1, ..., n\}$ compute $\langle \pi_X, \pi_X \rangle$ and decompose π_X to a direct sum of irreducible sub representations.

2. Construct representative for each class in $Irr(S_3)$.

3. Let $G = A_4$.

   (a) Consider the standard thetrahedron $\mathcal{T} \subset \mathbb{R}^3$. Denote by $(,)$ the usual inner product on $\mathbb{R}^3$ and by $SO(3) = \{A \in GL_3(\mathbb{R});\ A \text{ preserves } (,) \text{ and } \det(A) = 1\}$. Consider the group $T = Stab_{SO(3)}(\mathcal{T})$. Write down a natural isomorphism

$$r : T \xrightarrow{\sim} A_4.$$

   (b) Compute $\#Irr(G)$. Compute $\dim(\sigma)$ for every $\sigma \in Irr(G)$.

   (c) Construct representative for each class in $Irr(G)$.

4. Let $G = S_4$.

(a) Consider the standard cube $\mathcal{C} \subset \mathbb{R}^3$. Consider the group $C = Stab_{SO(3)}(\mathcal{C})$. Write down a natural isomorphism

$$r : C \widetilde{\to} S_4.$$

(b) Compute $\#Irr(G)$. Compute $\dim(\sigma)$ for every $\sigma \in Irr(G)$.

(c) Find two one dimensional irreps of G.

(d) Consider the permutation representation π of G on $\mathcal{H} = \{1, 2, 3, 4\}$ and obtain a three-dimensional irreducible representation of G.

(e) Consider the sign representation $sgn : S_n \to \{\pm\}$. Show that the representation $\rho = sgn \otimes \pi$ are not isomorphic.

(f) Denote by $Y = \{P_1, P_2, P_3\}$ the set of pairs of antipodal faces of the cube $\mathcal{C}$. Then G acts on Y. Use this to construct a new irreducible representation τ of G.

(g) Construct representative for each class in $Irr(G)$.

5. Let $G = A_5$.

(a) Consider the standard dodecahedron $\mathcal{D} \subset \mathbb{R}^3$. Consider the group $D = Stab_{SO(3)}(\mathcal{D})$. Write down a natural isomorphism

$$r : D \widetilde{\to} A_5.$$

(b) Write down the conjugacy classes of D and compute $\#Irr(G)$.

(c) Denote by ρ_1 the trivial representation of G. Denote by ρ_3 the tautological three dimensional representation of D, and show that is is irreducible.

(d) Compute $\dim(\sigma)$ for every $\sigma \in Irr(G)$.

(e) Use the permutation representation of A_5 on $X = \{1, 2, 3, 4, 5\}$ to construct a four dimensional irreducible representation ρ_4.

(f) Denote by $Y = \{A_1, ..., A_6\}$ the set of six axes through the centers of opposite centers of $\mathcal{D}$. Denote by π_Y the associated permutation representation. Compute the intertwining number $\langle \rho_1, \pi_Y \rangle$ and construct a five dimensional irreducible representation ρ_5 of D.

(g) Take an element $\theta \in S_5 \setminus A_5$. Consider the induced (by conjugation) automorphism $\alpha_\theta : A_5 \to A_5$. Define $\rho_3^\theta = \rho_3 \circ \alpha_\theta$. Show that ρ_3^θ and ρ_3 are not isomorphic.

(h) Construct representative for each class in $Irr(G)$.

Good Luck!