Introduction to Representation Theory
HW0 - Fall 2016
For Thursday Sep. 15 Meeting

Unless stated otherwise, all spaces are over $\mathbb{C}$ and are of finite dimension, and all groups are finite.

1. *Intertwiners.* Let (ρ, V) be a representation of a group G.

   (a) Define the natural action of G on the space $Hom(V, V)$ of all linear transformations on V.

   (b) For a space W with an action of a group G we can consider the space of <u>invariant vectors</u>

   $$W^G = \{w \in W;\ gw = w\}.$$

   Show that

   $$Hom_G(V, V) = Hom(V, V)^G,$$

   where $Hom_G(V, V) = \{\varphi : V \to V;\ \varphi \text{ is an intertwiner of } \rho\}$.

2. *Complete reducibility.*

   (a) Recall the notion of <u>sub-representation</u> of a representation (ρ, G, V). Recall when a representation of a group G is called <u>irreducible</u>.

   (b) Recall when we say that a representation is equal to a <u>direct sum</u> of representations.

   (c) Quote Maschke's theorem on <u>complete reducibility</u> of (ρ, G, V).

   (d) Prove Maschke's theorem.

3. *Schur's lemma.*

   (a) Prove the following form of Schur's lemma. Let τ and π be two irreducible representations of a group G. Show that

   $$\dim Hom(\tau, \pi) = \begin{cases} 1, & \text{if } \tau \simeq \pi; \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Suppose $(\rho, V) = (\rho_1, V_1) \oplus (\rho_2, V_2)$ is a decomposition of ρ into irreducible representations and ρ_1 and ρ_2 are not isomorphic. Show that any intertwiner $\varphi \in Hom(\rho, \rho)$ maps V_i to V_i, $i = 1, 2$.

4. *Special orthogonal group.* Consider the group $G = SL_2(\mathbb{F}_q)$ with q a power of an odd prime number p. Consider the standard Weyl element

   $$\mathrm{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that the group

$$C_G(\mathrm{w}) = \{g \in G;\ g\mathrm{w} = \mathrm{w}g\},$$

is commutative. Can you show it is cyclic?

(b) Compute $\#C_G(\mathrm{w}) = ?$.

(c) Consider the following symmetric non-degenerate bilinear form $\langle , \rangle$ on $\mathbb{F}_q^2$:

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle = xx' + yy'.$$

Show that $C_G(\mathrm{w}) = SO(\langle , \rangle) = \{g \in SL_2(\mathbb{F}_q);\ \langle gu, gv \rangle = \langle u, v \rangle$ for every $u, v \in \mathbb{F}_q^2\}$.

5. *Bruhat decomposition.* Consider the standard Borel subgroup of $G = SL_2(\mathbb{F}_q)$

$$B = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix};\ a \neq 0\}.$$

Show that the Bruhat decomposition holds

$$G = B \bigsqcup B\mathrm{w}B,$$

where w is the Weyl element.

6. *Conjugacy classes.* For q odd, verify that a list of $q+4$ elements representing all the conjugacy classes of $G = SL_2(\mathbb{F}_q)$, can be chosen from

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & \epsilon \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} x & y\epsilon \\ y & x \end{pmatrix},$$

where $\epsilon \in \mathbb{F}_q^*$ is a non-square, a runs in some natural subset of $\mathbb{F}_q^*$, and (x, y) runs over an appropriate subset of $\mathbb{F}_q \times \mathbb{F}_q$.

Good Luck!