Math 843
Representation Theory, Fall 11 – Problems Set 4
Character of permutation representation, $Irr(S_4)$

October 10, 2011

1. **Character of permutation representations.**

   Definition. Let (ρ, G, V) be a finite-dimensional representation a finite group G. We define the character of ρ to be the function

$$\begin{aligned} \chi_\rho &: G \to \mathbb{C}, \\ \chi_\rho(g) &= Trace[\rho(g)]. \end{aligned}$$

   (a) Show that if $\rho \simeq \pi$ representations of G then $\chi_\rho = \chi_\pi$.
   (b) Show that for any two representations ρ, π of the group G we have $\chi_{\rho \oplus \pi} = \chi_\rho + \chi_\pi$.
   (c) Let X be a G-set, and consider the associated permutation representation $(\Pi_X, G, \mathbb{C}(X))$. Show that for an element $g \in G$ we have

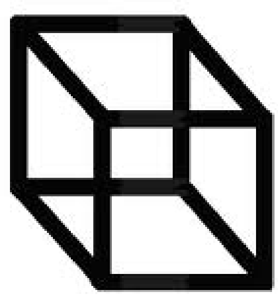
$$\chi_{\Pi_X}(g) = \#X^G = \text{number of fixed points for } G \text{ in } X.$$

2. **Irreducible representation of the group** $G = S_4$.

   (a) The group $G = S_4$ of all permutations of four objects has two 1-dim irreducible representations the trivial **1** and the sign representation *sgn*.
   (b) Consider the action of G on the set $X = \{a, b, c, d\}$, and the associated permutation representation $(\Pi_X, G, \mathbb{C}(X))$. We have a decomposition

$$\mathbb{C}(X) = \mathcal{H}_c \oplus \mathcal{H}_0,$$

   the direct sum of constant and mean zero functions. Show that $\mathcal{H}_0$ is a 3-dim irreducible representation.
   (c) The group G is isomorphic to the octahedral group $O \subset SO(3)$ of rotational symmetries of the cube C



   As a result we have an action of the group G on the set Y of the three pairs of parallel faces, and hence a permutation representation $(\Pi_Y, G, \mathbb{C}(Y))$. We have a decomposition

$$\mathbb{C}(Y) = \mathcal{H}_c \oplus \mathcal{H}_0,$$

   the direct sum of constant and mean zero functions. Show that $\mathcal{H}_0$ is a 2-dim irreducible representation.
   (d) Consider the 3-dim representation of G given by $\rho = sgn \cdot \Pi_X$ on $\mathbb{C}(X)$ via the formula $[sgn \cdot \Pi_X](g) = sgn(g) \cdot \Pi_X(g)$. Show that ρ is not isomorphic to Π_X.
   (e) Find irreducible representations realizing all the classes in $Irr(S_4)$.

Good Luck