Math 843
Representation Theory, Fall 11 – Problems Set 3
Weil representation, intertwining numbers, $Irr(S_3)$, $Irr(A_4)$

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1. **Construction of the Weil representation of** $SL_2(k)$, $k = \mathbb{F}_p$, $p \neq 2$.

   (a) Let (V, ω) be the two-dimensional symplectic vector space over with $V = k \times k$, and $\Omega : V \times V$, given by $\omega[(t, w), (t', w')] = tw' - w't$. Consider the Heisenberg group with the set $H = V \times k$ and the multiplication law given by $(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\Omega(v, v'))$. Show that the action $\gamma : SL_2(k) \times H \to H$ given by $\gamma[g](v, z) := ((g^{-1}v, z)$ is such that every $\gamma(g) : H \to H$ is a homomorphism. In addition show that the center of H is $Z(H) = \{(0, z);\ z \in k\}$. We will denote it simply by Z

   (b) Show that if (ρ, G, U) is irreducible representation, then we have $\rho(g) = \chi(g) \cdot Id_U$, for every $g \in Z(G)$–the center of G– where $\chi : Z(G) \to \mathbb{C}^*$ is a homomorphism.

   (c) Prove the Stone–von Neumann: Let $(\pi_j, H, \mathcal{H}_j)$, $j = 1, 2$, be two irreducible representation of the Heisenberg group H. If $\pi_j(z) = \psi(z) \cdot Id_{\mathcal{H}_j}$, $j = 1, 2$, with $\psi \neq 1$, then $\pi_1 \simeq \pi_2$.

   (d) We fix a non-trivial additive character $\psi : Z \to \mathbb{C}^*$ (e.g. $\psi(z) = e^{\frac{2\pi i}{p}z}$). Consider the associated Heisenberg representation on the space $\mathcal{H} = \mathbb{C}(k)$ of complex valued functions on k, given by

   $$\begin{aligned} \pi &: H \to GL(\mathcal{H}), \\ [\pi(t, w, z)f](x) &= \psi(\frac{1}{2}tw + z) \cdot \psi(wx) \cdot f(x + t). \end{aligned}$$

      i. Show that for every $g \in SL_2(k)$ the representations $(\pi, \mathcal{H})$, $({}^g\pi, H, \mathcal{H})$, where ${}^g\pi(v, z) = \pi(g^{-1}v, z)$ are isomorphic, and that $\dim Hom(\pi, {}^g\pi) = 1$. Deduce that there exists a collection of operators $\{\rho(g) \in GL(\mathcal{H});\ g \in SL_2(k)\}$ which satisfy

      $$\rho(g_1) \circ \rho(g_2) = c(g_1, g_2) \cdot \rho(g_1 g_2),$$

      and

      $$\rho(g) \circ \pi(h) = \pi(\gamma[g](h)) \circ \rho(g), \tag{1}$$

      for every $g, g_1, g_2 \in SL_2(k)$, $h \in H$, where $c(g_1, g_2) \in \mathbb{C}^*$ is a scalar which depends on g_1, g_2.

      Theorem (Schur). There exists a (unique if $p > 3$) representation

      $$\rho : SL_2(k) \to GL(\mathcal{H}),$$

      which satisfies the identity (1). This representation is called the Weil representation.

   (e) Using the identity (1) show that there exist constants C_b, C_a, C_{w} such that

   - $\left[\rho\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} f\right](x) = C_b \cdot \psi(-\frac{1}{2}bx^2) \cdot f(x);$
   - $\left[\rho\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f\right](x) = C_a \cdot f(a^{-1}x);$
   - $\left[\rho\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right](x) = C_{\mathrm{w}} \cdot \frac{1}{\sqrt{p}} \sum\limits_{y \in k} \psi(yx) \cdot f(x),$

   for every $b, x \in k$, $a \in k^*$, $f \in \mathcal{H}$.

   Remark. One can show that $C_b = 1$, $C_a = \left(\frac{a}{p}\right)$ =the Legendre symbol of a, and $C_{\mathrm{w}} = i^{\frac{p-1}{2}}$.

   (f) Show that $SL_2(k) = B \bigsqcup B\mathrm{w}B$ – the Bruhat decomposition, where B is the group of lower triangular matrices in $SL_2(k)$ and $\mathrm{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Conclude that (for $p > 3$) the above formulas characterize the model $(\rho, SL_2(k), \mathcal{H})$ that satisfies the identity (1).

2. **Intertwining numbers between permutation representations.** Let X, Y be two G-sets. Consider the associated permeation representations $(\Pi_X, G, \mathbb{C}(X))$, and $(\Pi_Y, G, \mathbb{C}(Y))$. We have a diagonal action of G on $X \times Y$ given by $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Use the steps below to prove the following:

   Proposition. We have

$$\langle \Pi_X, \Pi_Y \rangle \overset{\text{def}}{=} \dim Hom(\Pi_X, \Pi_Y) = \# \ (X \times Y)/G - \text{number of orbits of } G \text{ in } X \times Y.$$

   (a) Define the natural action of G on the vector space $Hom(\mathbb{C}(X), \mathbb{C}(Y))$ of linear operators from $\mathbb{C}(X)$ to $\mathbb{C}(Y)$. Define the natural action of G on the vector space of functions $\mathbb{C}(X \times Y)$.

   (b) Verify that we have a canonical isomorphism of representations of G

$$\begin{array}{rcl} K & : & Hom(\mathbb{C}(X), \mathbb{C}(Y)) \tilde{\rightarrow} \mathbb{C}(X \times Y), \\ \alpha & \longmapsto & K_\alpha, \end{array} \tag{2}$$

   sending a linear operator α to its <u>kernel</u> function (matrix with respect to delta basis) K_α given by

$$\alpha(\delta_x) = \sum_{y \in Y} K_\alpha(x, y) \cdot \delta_y, \quad x \in X, y \in Y,$$

   where $\{\delta x\}$, $\{\delta y\}$ are the natural bases of delta functions on $\mathbb{C}(X)$, and $\mathbb{C}(Y)$ respectively.

   (c) For a representation (ρ, G, V) we can attach the natural vector space $V^G = \{v \in V; \rho(g)v = v$ for every $g \in G\}$ of G-<u>invariant vectors</u> in V. Show that if (ρ_j, G, V_j), $j = 1, 2$, and $I : V_1 \to V_2$ is an intertwiner, then we have an induced isomorphism $I^G : V_1^G \tilde{\rightarrow} V_2^G$.

   (d) Show that the kernel morphism (2) induces an isomorphism $K^G : Hom(\mathbb{C}(X), \mathbb{C}(Y))^G \tilde{\rightarrow} \mathbb{C}(X \times Y)^G$, which concretely means that $\alpha \in Hom(\mathbb{C}(X), \mathbb{C}(Y))$ is an intertwiner iff $K_\alpha(g \cdot x, g \cdot y) = K_\alpha(x, y)$ for every $x \in X$, $y \in Y$, $g \in G$. To conclude we obtained that

$$\begin{aligned} \langle \Pi_X, \Pi_Y \rangle &= \dim Hom(\Pi_X, \Pi_Y) \\ &= \dim Hom_G(\mathbb{C}(X), \mathbb{C}(Y)) \\ &= \dim Hom(\mathbb{C}(X), \mathbb{C}(Y))^G \\ &= \dim \mathbb{C}(X \times Y)^G \\ &= \# \ (X \times Y)/G. \end{aligned}$$

3. **Irreducible representations of** $G = S_3$.

   (a) The group G acts naturally on the equilateral triangle $\triangle$ with vertices a, b, c. We have a natural action of G on the sets $X_0 = \{pt\}$– trivial action on one point, and $X_1 = \{a, b, c\}$. Compute the intertwining numbers table

$$\left\langle \Pi_{X_i}, \Pi_{X_j} \right\rangle_{0 \le i,j \le 1} = \begin{array}{ccc} & \Pi_{X_0} & \Pi_{X_1} \\ \Pi_{X_0} & ? & ? \\ \Pi_{X_1} & ? & ? \end{array} \tag{3}$$

   (b) Consider the natural decomposition of the representation $\mathbb{C}(X_1) = \mathcal{H}_c \oplus \mathcal{H}_0$ into the direct sum of the spaces of constant functions on the vertices, and functions with average $= 0$ on the vertices. Deduce from (3) that this is a decomposition into irreducible representations.

   (c) Write down models for all the irreducible representations of the group G.

4. **Irreducible representations of** $T = A_4$. Consider the Thetrahedral group of rotational symmetries of the following Thetrahedron PWe would like to find representations of T which realizes all of $Irr(T)$.

   (a) Consider the set $Y = \{(e_{1\leftrightarrow 3}, e_{2\leftrightarrow 4}), (e_{3\leftrightarrow 2}, e_{4\leftrightarrow 1}), (e_{4\leftrightarrow 3}, e_{1\leftrightarrow 2})\}$ of pairs of non coplanar edges in T, where $e_{i\leftrightarrow j}$ denotes the edge between vertices i and j. Verify that the group T acts on the set Y and this means that we have an homomorphism

$$r : T \to Aut(Y).$$

   However, this morphism is not onto. Consider the kernel $K = \ker(r)$.

i. Compute explicitly the group K. In particular, write down an explicit projection $T \twoheadrightarrow T/K = C_3$ cyclic group of size three.

ii. Find three 1-dimensional representations of T.

(b) Consider the set X =vertices of P. Verify that the associated permutation representation $(\Pi_X, T, \mathbb{C}(X))$ decomposes into the direct sum

$$\mathbb{C}(X) = \mathcal{H}_c \oplus \mathcal{H}_0,$$

of 1-dim invariant subspace of constant functions, and 3-dim invariant subspace of functions f whose sum $\sum_{x \in X} f(x) = 0$. Compute the intertwining number $\langle \Pi_X, \Pi_X \rangle =?$, and deduce that $\mathcal{H}_0$ is irreducible representation.

(c) Find representatives for all the classes in $Irr(T)$.

Good Luck