HOMEWORK3

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1. PROBLEM2:INTERTWINING NUMBER BETWEEN PERMUTATION PREPRESENTATIONS

1.1. **(a).** G acts on $Hom(\mathbb{C}(X), \mathbb{C}(Y))$ by $\forall g \in G$, $\forall T \in Hom(\mathbb{C}(X), \mathbb{C}(Y))$

$$(g \cdot T)(a) := g \cdot (T(g^{-1}a)) \qquad \forall\ a \in\ \mathbb{C}(X).$$

i.e.

$$g \cdot T := \Pi_Y(g) \circ T \circ \Pi_X(g^{-1}).$$

G acts on $\mathbb{C}(X \times Y)$ by $\forall g \in G$,$\forall K \in \mathbb{C}(X, Y)$

$$(g \cdot K)(x, y) := K(g \cdot x, g \cdot y)$$

1.2. **(b).** WTS: K commute with G action, i.e. $\forall g \in G$, $K(g \cdot \alpha) = g \cdot K(\alpha)$. More concretely, $\forall x$, $\forall y$, $K(g \cdot \alpha)(x, y) = g \cdot K(\alpha)(x, y)$. On one side we have:

$$\begin{aligned}
(g \cdot \alpha)(\delta_x) \qquad &= g \cdot \alpha(g^{-1} \cdot \delta_x) \\
&= g \cdot (\alpha(\delta_{g^{-1}x})) \\
&= g \cdot (\sum_{y \in Y} K_\alpha(g^{-1} \cdot x, y)\delta_y) \\
&= \sum_{y \in Y} K_\alpha(g^{-1} \cdot x, y) g \cdot \delta_y \\
&= \sum_{y \in Y} K_\alpha(g^{-1} \cdot x, y)\delta_{g \cdot y} \\
&= \sum_{y \in Y} K_\alpha(g^{-1} \cdot x, g^{-1} \cdot y)\delta_y \\
&= \sum_{y \in Y} g \cdot K_\alpha(x, y)\delta_y
\end{aligned}$$

On the other side

$$(g \cdot \alpha)(\delta_x) = \sum_{y \in Y} K_{g \cdot \alpha}(x, y)\delta_y$$

By comparing coefficient of δ_y, we have $K_{g \cdot \alpha}(x, y) = g \cdot K_\alpha(x, y)$.

1.3. **(c).**

(1) Well-defined: $\forall a \in V_1^G$ need to show $g \cdot I(a) = I(a)$. This is because of I commute with G action.
(2) injective: I is injective

(3) surjective: $\forall b \in V_2^G$, $a := I^{-1}(b)$, NTS: $g \cdot a = a$. Note that $I(g \cdot a) = g \cdot I(a) = g \cdot b = b = I(a)$, since I is injective, $g \cdot a = a$.

1.4. **(d).**

$$\begin{aligned}
< \Pi_X, \Pi_Y > &= dimHom(\Pi_X, \Pi_Y) \\
&= dimHom_G(\mathbb{C}(X), \mathbb{C}(Y)) && intertwiner = isomorphism\ commute\ with\ G \\
&= dimHom(\mathbb{C}(X), \mathbb{C}(Y))^G && T \circ \Pi_X(g) = \Pi_Y(g)T \Rightarrow T = \Pi_Y(g)T \circ T \circ \Pi_X(g)^{-1} \\
&= dim\mathbb{C}(X, Y)^G && (b), (c) \\
&= \#(X \times Y)/G
\end{aligned}$$

The last identity is because for each orbit $G \cdot (x, y)$ one can define

$$K_{G\cdot(x,y)} := \sum_{y \in G} \delta_{g\cdot x, g\cdot y}$$

$\{K_{G\cdot(x,y)}\}$ form a base of $\mathbb{C}(X, Y)^G$, since any function in $\mathbb{C}(X, Y)^G$ has to be a constant over an orbit of G.

2. Irreducible representation of S_3

2.1. **(a).**

$$< \Pi_{X_0}, \Pi_{X_0} >= \#(\{pt\} \times \{pt\})/S_3 = 1$$

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$$< \Pi_{X_1}, \Pi_{X_1} >= \#(X_1 \times X_1)/S_3 = \#\{\Delta_{X_1}, X_1 \times X_1 \setminus \Delta_{X_1}\} = 2$$

Now we show the second last identity $X_1 \times X_1)/S_3 = \{\Delta_{X_1}, X_1 \times X_1 \setminus \Delta_{X_1}\}$:

$\forall (a, b) \neq (a', b') \in X_1 \times X_1 \setminus \Delta_{X_1}$, without loss of generality assume $a \neq a'$, $a \neq b$ and $a' \neq b'$ since they are off diagonal. Now depending on $b = a'$ or not we have two cases:

- $a' = b$. In this case b' is either different from $b = a'$ and a, ie $\{a, b, b'\} = X_1$ or $b' = a$:
  - $\{a, b, b'\} = X_1$: $(abb') \cdot (a, b) = (b, b') = (a', b')$;
  - $b' = a$: $(ab) \cdot (a, b) = (b, a) = (a', b')$.
- $a' \neq b$. In this case b' is either a or b:
  - $b' = a$: $(baa') \cdot (a, b) = (a', a) = (a', b')$;
  - $b' = b$: $(aa') \cdot (a, b) = (a', b) = (a', b')$.

2.2. **(b).** $\mathcal{H}_c$ and $\mathcal{H}_0$ are S_3-invariant.

$\Pi|_{\mathcal{H}_c}$ is irreducible since $\mathcal{H}_c$ is 1-dim.

$$\begin{aligned}
2 =< \Pi_{X_1}, \Pi_{X_1} > &=< \Pi|_{\mathcal{H}_c} \oplus \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_c} \oplus \Pi|_{\mathcal{H}_0} > \\
&=< \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_c} > +2 < \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_0} > + < \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_0} > \\
&= 1 + 2 < \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_0} > + < \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_0} >
\end{aligned}$$

This force $< \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_0} >= 0$ and $< \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_0} >= 1$ So $\Pi|_{\mathcal{H}_0}$ is an irreducible representation.

2.3. **(c).** By $\#G = \sum_{\pi\in Irr(G)}(dim\pi)^2$:

$$6 = \#S_3 = 1^2 + 2^2 + 1^2$$

Where the first 1-dim irreducible representation is $\Pi|_{\mathcal{H}_c}$(the trivial representation), the 2-dim irreducible representation $\Pi|_{\mathcal{H}_0}$. Only need to find the other 1-dim representation. Note that we have a character $\rho : S_3 \to \mathbb{Z}_2$ which send odd permutations to -1 and even permutations to 1. This defines a nontrivial 1-dim irreducible representation.

3. Irreducible representation of A_4

3.1. **(a).** A_4 acts on Y by $\forall\sigma \in A_4$, $\forall(e_{a\leftrightarrow b}, e_{c\leftrightarrow d}) \in Y$, $\sigma \cdot (e_{a\leftrightarrow b}, e_{c\leftrightarrow d}) = (e_{\sigma(a)\leftrightarrow\sigma(b)}, e_{\sigma(c)\leftrightarrow\sigma(c)})$. Note that $(e_{\sigma(a)\leftrightarrow\sigma(b)}, e_{\sigma(c)\leftrightarrow\sigma(c)}) \in Y$ since $(e_{\sigma(a)\leftrightarrow\sigma(b)}, e_{\sigma(c)\leftrightarrow\sigma(c)})$ are non-coplanar as far as $\{\sigma(a), \sigma(b)\} \neq \{\sigma(c), \sigma(d)\}$

i) compute K:

If $\sigma \in K$ then $\forall(e_{a\leftrightarrow b}, e_{c\leftrightarrow d}) \in Y$, $\sigma \cdot (e_{a\leftrightarrow b}, e_{c\leftrightarrow d}) = (e_{\sigma(a)\leftrightarrow\sigma(b)}, e_{\sigma(c)\leftrightarrow\sigma(c)}) = (e_{a\leftrightarrow b}, e_{c\leftrightarrow d})$. This means σ must be one of the followings:

- $\sigma(a) = a$, $\sigma(b) = b$,$\sigma(c) = c$, $\sigma(d) = d$, so $\sigma = id$;
- $\sigma(a) = b$, $\sigma(b) = a$,$\sigma(c) = d$, $\sigma(d) = c$, so $\sigma = (ab)(cd)$;
- $\sigma(a) = c$, $\sigma(b) = d$,$\sigma(c) = a$, $\sigma(d) = b$, so $\sigma = (ac)(bd)$;
- $\sigma(a) = d$, $\sigma(b) = c$,$\sigma(c) = b$, $\sigma(d) = a$, so $\sigma = (ad)(bc)$.

$\pi : A_4 \to A_4/K = \mathbb{Z}_3$ is determined by $\pi((ab)(cd)) = 1$ for any $(ab)(cd) \in K$, $\pi((abc)) = e^{\frac{2\pi i}{3}}$ for (abc)=(123),(243),(134),(142), and $\pi((abc)) = e^{\frac{4\pi i}{3}}$ for the rests.

i) find three 1-dim representation of A_4 We have three characters by postcomposing π with $\rho_k : \mathbb{Z}_3 \to \mathbb{C}^*$, where for $k = 0, 1, 2$, $\rho_i(e^{\frac{2\pi i}{3}}) := e^{\frac{2k\pi i}{3}}$.

3.2. **(b).** We first show that

$$X \times X)/A_4 = \{\Delta_X, X \times X \setminus \Delta_X\}.$$

$\forall(a, b) \neq (a', b') \in X \times X \setminus \Delta_X$, we have the following cases:

- a, b, a', b' are all different: $(aa')(bb') \cdot (a, b) = (a', b')$;
- There are three distinct elements among a, b, a', b': use cycle of the three elements;
- $a' = b$, $b' = a$: $(ab)(cd) \cdot (a, b) = (b, a) = (a', b')$

Thus

$$< \Pi_X, \Pi_X >= \#(X \times X)/A_4 = \#\{\Delta_X, X \times X \setminus \Delta_X\} = 2$$

$$\begin{aligned} 2 =< \Pi_X, \Pi_X > \quad &=< \Pi|_{\mathcal{H}_c} \oplus \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_c} \oplus \Pi|_{\mathcal{H}_0} > \\ &=< \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_c} > +2 < \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_0} > + < \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_0} > \\ &= 1 + 2 < \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_0} > + < \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_0} > \end{aligned}$$

This force $< \Pi|_{\mathcal{H}_c}, \Pi|_{\mathcal{H}_0} >= 0$ and $< \Pi|_{\mathcal{H}_0}, \Pi|_{\mathcal{H}_0} >= 1$ So $\Pi|_{\mathcal{H}_0}$ is an irreducible representation.

3.3. **(d).**

$$12 = 1^2 + 1^2 + 1^2 + 3^2$$

The representation of 3-dim is $\Pi|_{\mathcal{H}_0}$, Three 1-dim representations are given in (a). Thus we have all the irreducible representations.

4. PROBLEM4 OF HOMEWORK 2:COMPLETE REDUCIBILITY AND FINITE FOURIER TRANSFORMATION

4.1. **(a).** They are determined by the characters $\phi_k : A \to C^*$, where for $k = 0...p-1$, $\phi_k(1) = e^{\frac{2k\pi i}{p}}$.

4.2. **(b).** $\mathcal{H}_\phi := \{c\phi | c \in \mathbb{C}\}$.

Orthogonal:

$$< \phi_k, \phi_j >= \frac{1}{|A|} \sum_{x \in A} \phi_k(x) \bar{\phi_j}(x) = \frac{1}{|A|} \sum_{x \in A} \phi_k(x) \phi_j(-x)$$

If $k = j$, $< \phi_k, \phi_j >= \frac{1}{|A|} \sum_{x \in A} \phi_k(x - x) = 1$;

If $k \neq j$,

$$< \phi_k, \phi_j >= \frac{1}{|A|} \sum_{x \in A} \phi_i(x) \phi_j(-x) = \frac{1}{|A|} \sum_{x \in A} e^{\frac{2(x \cdot k - x \cdot j)\pi i}{p}} = \frac{1}{|A|} \sum_{x \in A} e^{\frac{2x(k-j)\pi i}{p}} = 0$$

Now $\mathcal{H} = \oplus_\phi \mathcal{H}_\phi$ by dimension counting.

$\mathcal{H}_{\phi_k}$ is A-invariant under Π_A: $[\Pi_A(y)\phi_k](x) = \phi_k(x + y) = \phi_k(y)\phi_k(x) = e^{\frac{2ky\pi i}{p}} \phi_k(x) \in \mathcal{H}_{\phi_k}$.

4.3. **(c).** Verify $P_{\phi_k}[f] \in \mathcal{H}_{phi_k}$:

$$\begin{aligned}
< P_{\phi_k}[f], \phi_j > &= \frac{1}{|A|} \sum_{x \in A} P_{\phi_k}[f](x) \phi_j(-x) \\
&= \frac{1}{|A|} \sum_{x \in A} \sum_{y \in A} \phi_k(-y) f(x + y) \phi_j(-x) \\
&= \frac{1}{|A|} \sum_{x, y \in A} \phi_k(-y - x) f(x + y) \phi_k(x) \phi_j(-x) \\
&= \frac{1}{|A|} \sum_{x \in A} \sum_{y \in A} \phi_k(-y) f(y) \phi_k(x) \phi_j(-x) \\
&= \sum_{y \in A} \phi_k(-y) f(y) \frac{1}{|A|} \sum_{x \in A} \phi_k(x) \phi_j(-x) \\
&= \sum_{y \in A} \phi_k(-y) f(y) < \phi_k, \phi_j >
\end{aligned}$$

If $k \neq j$

$$< P_{\phi_k}[f], \phi_j >= 0$$

If k=j

$$< P_{\phi_k}[f], \phi_k >= \sum_{y \in A} \phi_k(-y) f(y)$$

Thus $P_{\phi_k}[f] \in \mathcal{H}_{phi_k}$. Moreover $P_{\phi_k}[f] = \sum_{k=0,p-1} < f, \phi_k > \phi_k$.(answer a question in (d)) P_ϕ is an orthogonal projection because its image is $\mathcal{H}_\phi$, kernel is $\oplus_{\phi' \neq \phi} \mathcal{H}_{\phi'}$, the two are orthogonal as a result of (b).

4.4. **(d).** Done in (b) and (c)