Math 843
Representation Theory, Fall 11 – Problems set 2
Schur's lemma, complete reducibility

October 1, 2011

1. **Schur's lemma.**

   (a) Recall the definition of irreducible representation.

   (b) State Schur's lemma about the intertwining space $Hom(\rho, \pi)$ for two irreducible representations (ρ, V_ρ) and (π, V_π) of a finite group G.

   (c) Let (ρ, V_ρ) and (π, V_π) be two irreducible representations of a finite group G. Consider the direct sum (τ, V). The space V has four G-invariant "coordinate" subspaces: 0, V_ρ, V_π, V. Show that the representations ρ and π are isomorphic iff there exists a non-coordinate G-invariant subspace in V (i.e., a subspace distinct from the four subspaces listed above).

2. **Maschke's theorem and unitary structure.**

   (a) Let (ρ, G, V) be an irreducible representation of a finite group G. Show that there exists a non-degenerate, positive definite, skew-symmetric bilinear form (i.e., a scalar product) $\langle \ , \ \rangle$ on V, which is G-invariant, i.e., such that $\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$ for every $u, v \in V$, and every $g \in G$. Such a scalar product is called G-invariant unitary structure. This mean that we can think on ρ as a unitary representation, i.e., a homomorphism $\rho : G \to U(V, \langle , \rangle)$ into the unitary group.

   (b) Use the fact that a representation (ρ, G, V) has a G-invariant unitary structure, to suggest an alternative proof to Maschke's theorem that every G-invariant subspace $W \subset V$ has a G-invariant complement.

   (c) Show that a representation (ρ, G, V) with a G-invariant unitary structure $\langle \ , \rangle$, is isomorphic to a orthogonal direct sum decomposition

   $$V \simeq \bigoplus_{\pi \in Irr(G)} m_\pi \cdot V_\pi,$$

   i.e., any two direct summands in your decomposition are orthogonal.

3. **Multiplicity free and diagonalization.**

   Definition Let (ρ, G, V) be a representation. If we have

   $$V \simeq \bigoplus_{\pi \in Irr(G)} m_\pi \cdot V_\pi,$$

   with $m_\pi = 1$ for every π that appears in the decomposition, then we say that ρ is multiplicity free.

   (a) Show that if ρ is multiplicity free, then any intertwiner $\alpha \in Hom(\rho, \rho)$ is diagonalizable.

   (b) Let X be a G-set. Then we have a representation $(\Pi_X, G, \mathbb{C}(X))$ on the space of complex valued functions on X, given by the formula $[\Pi_X(g)f](x) = f(g^{-1}x)$, for every $g \in G$, $f \in \mathbb{C}(X)$, and $x \in X$. This representation is called permutation representation. Consider the triangle $\triangle = \triangle(a, b, c)$ with vertices a, b, c, and the associated representation $(\Pi_\triangle, S_3, \mathbb{C}(\triangle))$ of the symmetric group S_3 on the space of functions on the set of vertices of $\triangle$. Show that $\Pi_\triangle$ is multiplicity free, and write down the explicit decomposition into direct sum of irreducible representations.

   (c) Consider the following "discrete Laplacian" operator $\bigtriangledown : \mathbb{C}(\triangle) \to \mathbb{C}(\triangle)$, given by the formula $[\bigtriangledown f](y) = \sum_{x \sim y} f(x)$, where $x \sim y$ means "x is a neighbor of y". Show that $\bigtriangledown$ is diagonalizable and compute its eigenvalues, and an orthonormal basis of eigenfunctions.

   (d) Use the facts that for a finite group G we have $\#Irr(G) = \#C.C.(G)$ the number of conjugacy classes in G, and that $\#G = \sum_{\pi \in Irr(G)} \dim(\pi)^2$ to find models (explicit realizations) for all the irreducible representations of $G = S_3$.

4. **Complete reducibility and the finite Fourier transform.**

(a) Find models (spaces with actions representing the equivalence classes) of all the irreducible representations of the additive group $A = (\mathbb{F}_p, +, 0)$.

(b) The abelian group $A = \mathbb{F}_p$ has a regular representation $(\Pi_A, A, \mathcal{H})$ on the space of functions $\mathcal{H} = \mathbb{C}(A)$, given by the formula $[\Pi_A(y)f](x) = f(x+y)$, for every $y \in G$, $f \in \mathcal{H}$, and $x \in A$. Show that we have a multiplicity free decomposition into orthogonal (we have a natural inner product structure on $\mathcal{H}$) direct sum decomposition of one dimensional spaces

$$\mathcal{H} = \bigoplus_{\psi : A \to \mathbb{C}^*} \mathcal{H}_\psi,$$

where ψ runs through the set of additive characters of $A = \mathbb{F}_p$, and $\mathbb{C}^*$ is multiplicative group of non-zero complex numbers.

(c) Given a function $f \in \mathcal{H}$ show how to decompose it into its components

$$f = \sum_{\psi : A \to \mathbb{C}^*} f_\psi, \quad f_\psi \in \mathcal{H}_\psi.$$

In particular, show that the operator

$$\begin{aligned} P_\psi &: \mathcal{H} \to \mathcal{H}, \\ P_\psi[f](x) &= \sum_{y \in A} \psi(-y) f(x+y), \end{aligned}$$

is the orthogonal projection onto the subspace $\mathcal{H}_\psi$. Hence, $P_\psi[f] = f_\psi$.

(d) Verify that in our case the character $\psi \in \mathcal{H}_\psi$. Hence we have another formula $P_\psi[f] = \langle f, \psi \rangle \cdot \psi$ to describe the orthogonal projector. Let us denote by $\widehat{A} =$ the "dual" group (with operation giving by multiplications) of additive characters on A. Then we obtained a transform $F : \mathbb{C}(A) \to \mathbb{C}(\widehat{A})$, given by the formula $F[f](\psi) = \langle f, \psi \rangle$. This is another (we saw one using the Heisenberg group in the motivation part of the course) description of the finite Fourier transform.

Good Luck