Math 843
Representation Theory, Fall 11 – Problems set 1
Symmetries of the platonic solids

September 18, 2011

1. *Action of a group on a set.*

   (a) Define the notion of a group G acting on a set X.

   (b) Denote the action of a group G on a set X by $\cdot$. Show that we have an induced homomorphism

   $$\begin{cases} \alpha : G \to Aut(X), \\ \alpha[g](x) = g \cdot x, \end{cases}$$

   where $Aut(X)$ denote the group of all bijections from X to itself.

   (c) Suppose that a group G acts on a set X and that $Y \subset X$ is a <u>G-invariant</u> subset of X, i.e., $g \cdot y \in Y$, for every $y \in Y$. Show that in this case we have an induced "restriction" homomorphism

   $$\begin{cases} r : G \to Aut(Y) \\ r(g)[y] = g \cdot y. \end{cases}$$

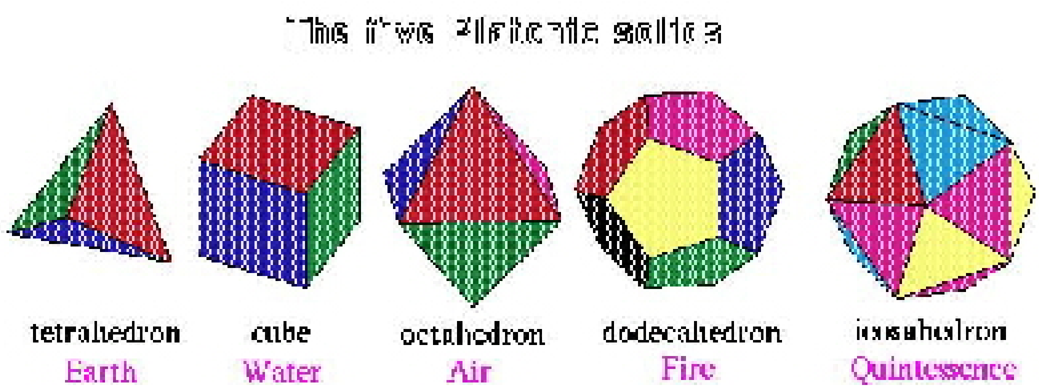
2. *The special orthogonal group $SO(3)$.*

   (a) Consider the vector space $V = \mathbb{R}^3$ with its standard inner product$\langle \ , \ \rangle$. Denote by $GL(V)$ the group of all invertible linear operators from V to itself. Denote by $O(3) = \{A \in GL(V);$ such that $\langle Au, Av\rangle = \langle u, v\rangle \ \forall \ u, v \in V\}$. Show that $O(3)$ is a subgroup of $GL(V)$. It is called the three-dimensional <u>orthogonal group.</u>

   (b) Show that for every $A \in O(3)$ we have $\det(A) \in \{-1, 1\}$.

   (c) Denote by denote by $SO(3) = \{B \in O(3); \det(B) = 1\}$. Show that $SO(3)$ is a <u>normal</u> (please recall what is this important notion) subgroup of $O(3)$. It is called the group of <u>rotations</u> of three-dimensional space, or the <u>special orthogonal group</u>.

   (d) We define the (rotational) <u>symmetries</u> of a subset $X \subset V$ to be the subgroup of $SO(3)$ given by the stabilizer $G = Stab_{SO(3)}(X) = \{B \in SO(3);\ B$ maps X onto itself$\}$. Show that G is indeed a subgroup of $SO(3)$.



3. *The symmetry group of the thetrahedron.* Denote by P the thetrahedron, which we place around the origin of $V = \mathbb{R}^3$. Consider the thetrahedral group $T = Stab_{SO(3)}(P)$.

   (a) Compute the cardinality $\#T =?$

   (b) Use the idea of subsection 2.c. to define an EXPLICIT homomorphism $r : T \to S_4 = Aut(\{a, b, c, d\})$. Show that your homomorphism is monomorphism (i.e., one-to-one). Hint: Four vectors given by the vertices of P.

(c) Show that if $H < S_n = Aut(\{a_1, ..., a_n\})$ is of index $[S_n : H] = 2$ then $H = A_n$ the subgroup of even permutations. Deduce that r induces an explicit isomorphism $T \simeq A_4$.

4. *The symmetry group of the cube.* Denote by C the cube, which we place around the origin of $V = \mathbb{R}^3$. Consider the octahedral group $O = Stab_{SO(3)}(C)$.

   (a) Compute the cardinality $\#O =$?

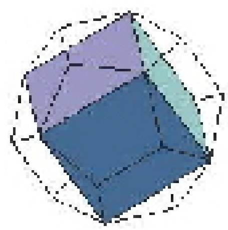
   (b) Use the idea of subsection 2.c. to define an EXPLICIT homomorphism $r : O \to S_4 = Aut(\{d_1, d_2, d_3, d_4\})$. Show that your homomorphism is monomorphism (i.e., one-to-one). Hint: Take $d_1, ..., d_4$ to be the diagonals of C.

   (c) Deduce that r induces an explicit isomorphism $O \simeq S_4$.

5. *The symmetry group of the dodecahedron.* Denote by D the Dodecahedron, which we place around the origin of $V = \mathbb{R}^3$. Consider the Icosahedral group $I = Stab_{SO(3)}(D)$.

   (a) Compute the cardinality $\#I =$?

   (b) Use the idea of subsection 2.c. to define an EXPLICIT homomorphism $r : I \to S_5 = Aut(\{C_1, C_2, C_3, C_4, C_5\})$. Show that your homomorphism is monomorphism (i.e., one-to-one). Hint: we can embed five cubes in a natural in the dodecahedron

   In addition, it is known (see the book Algebra by Michael Artin) that I is a simple group, i.e. has no non-trivial normal subgroups.

   (c) Deduce that r induces an explicit isomorphism $I \simeq A_5$.

Good Luck!