Symmetry Groups of the Platonic Solids

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In this seminar, we will determine the symmetry groups of the Platonic solids. Note that we need only consider the tetrahedron, cube, and dodecahedron, since the octahedron and icosahedron are duals of the cube and dodecahedron. We will follow a general pattern, in which we first describe a map from our desired symmetry group into a group of permutations, prove that this map is injective, and then find the order of the symmtery group using the Orbit-Stabilizer Theorem, and use it to deduce the image of said map.

Definition. We define the <u>symmetry group</u> of a platonic solid (or, in general, of a set in $\mathbb{R}^3$) to be the group of matrices in $SO(3)$ preserving the solid (or set).

Note that this definition does not include symmetries given by reflections; we only consider rotations in $\mathbb{R}^3$. Including all symmetries in $O(3)$ is also an interesting problem, but here we will only consider those in $SO(3)$.

1 The Tetrahedron

Let T denote the symmetry group of the tetrahedron. T acts on the set of vertices of the tetrahedron, and thus we have a map $\phi_T : T \to S_4$.

Claim 1.1. *The map ϕ_T is injective.*

Proof: If ϕ_T is not injective, then there is some element $t \in T$ besides the identity matrix which fixes all the vertices. However, pick any three of the vertices. These are three linearly independent vectors in $\mathbb{R}^3$, and t fixes them all. Thus t must be the identity matrix. This shows that the kernel of ϕ_T is trivial, so ϕ_T is injective.

This map gives T as a subgroup of S_4. We will now find the order of T.

Claim 1.2. $|T| = 12$.

Proof: Consider the action of T on the vertices of the tetrahedron. The action is transitive, so the orbit of any vertex has size 4. And the stabilizer of any vertex has size 3, since there are 3 rotations around that vertex that preserve T. Thus $|T| = 12$.

Now we know that T is isomorphic to an index-2 subgroup of S_4.

Claim 1.3. *The only index-2 subgroup of S_n is the alternating group A_n.*

Proof: Let H be an index-2 subgroup of S_n. H must be normal (because it has index 2), and so we get a nontrivial map $s : S_n \to S_n/H \simeq \{\pm 1\}$. Since the image of this map is an abelian group, each element of a conjugacy class must map to the same thing (since $s(\sigma\tau\sigma^{-1}) = s(\sigma)s(\sigma^{-1})s(\tau) = s(\tau)$); in particular, all transpositions must map to the same element of $\{\pm 1\}$. Since S_n is generated by transpositions, if they all map to 1, then s is trivial. So all transpositions must map to -1. But this completely determines s, and makes s the sign map, which has kernel A_n. Thus $H = A_n$, as desired.

This proves that T is isomorphic to the alternating group A_4.

2 The Cube

Let C be the symmetry group of the cube. The cube has four long diagonals (connecting opposite vertices), and C acts on them, so we have a map $\phi_C : C \to S_4$.

Claim 2.1. *The map ϕ_C is injective.*

Proof: As before, suppose there is an element $c \in C$ fixing all four diagonals, so for each pair of opposite vertices, c either fixes them or interchanges them. Suppose c does not fix every vertex. Then we can pick three pairs of opposite vertices so that c either fixes none of them, or fixes two of the pairs. (That is, c interchanges an odd number of the pairs.) Now pick one vertex from each pair, and use these three vertices as basis vectors for $\mathbb{R}^3$. When c is written in this basis, it becomes a diagonal matrix with either three -1's, or one -1 and two 1's on the diagonal. (The -1's correspond to vertex pairs that are interchanged by c.) But this makes c have determinant -1, which is not possible because c is in $SO(3)$. Thus c must in fact fix every vertex; in particular, it fixes three linearly independent vectors, and so is the identity matrix.

This shows that ϕ_C has trivial kernel. As before, we now have C as a subgroup of S_4, so we can compute its order.

Claim 2.2. $|C| = 24$.

Proof: As before, consider the action of C on the vertices of the cube. The orbit of any vertex has size 8, and the stabilizer has size 3. Thus by orbit-stabilizer, $|C| = 24$.

Since C is isomorphic to a subgroup of S_4, and $|C| = 24$, C must be isomorphic to S_4 itself.

3 The Dodecahedron

Let D be the symmetry group of the dodecahedron. The dodecahedron contains five cubes inside of it; each vertex of the dodecahedron is shared by two cubes, and each face diagonal of the dodecahedron is an edge of exactly one cube. D acts on these cubes, and so we get a map $\phi_D : D \to S_5$.

We will take a different approach this time to proving that ϕ_D is injective; we will do so by proving that D is simple, which requires some preliminaries.

Claim 3.1. *If $M \in SO(3)$, then M can be described as a rotation of $\mathbb{R}^3$ about some line; that is, M must have 1 as an eigenvalue.*

Proof: M must have a real eigenvalue λ (because its characteristic polynomial has odd degree). Choose any basis in which one of the basis vectors is an eigenvector for λ. In this basis, M looks like

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & & \\ 0 & & M' \end{pmatrix}$$

Here M' is some 2-by-2 matrix (which will depend on the basis chosen). Since M is orthogonal, this requires $\lambda = \pm 1$. If $\lambda = -1$, then the fact that $|M| = \lambda |M'| = 1$ implies that $|M'| = -1$. But then the characteristic polynomial of M' is a quadratic polynomial with negative constant term (namely $|M'|$), and such polynomials always have real roots. Thus M has all real eigenvalues. For the same reason as before, all the eigenvalues of M must be ± 1. But if they are all -1, then $|M| = -1$, which can't happen. Thus M has 1 as an eigenvalue, as desired.

Claim 3.2. $|D| = 60$.

Proof: Using orbit-stabilizer as above, consider the action of D on the vertices of the dodecahedron. The orbit of each vertex has size 20, and the stabilizer of each vertex has size 3, so $|D| = 60$.

Now we can get back to our main line of reasoning:

Claim 3.3. *The group D is simple.*

Proof: Because of the previous claim, each element of D is a rotation around some line in $\mathbb{R}^3$ through the origin (assuming we place the center of D at the origin). The only lines around which we can rotate nontrivially and preserve D are (1) lines through a vertex, (2) lines through the center of a face, and (3) lines through the midpoint of an edge.

All of the rotations of type (1) are conjugate to each other; there are 10 such lines (one for each pair of opposite vertices), and 2 nontrivial rotations per line, so this conjugacy class contains 20 elements.

Rotations of type (2) fall into two classes: (2a) rotations by 72 degrees (in either direction), and (2b) rotations by 144 degrees (in either direction). The rotations in each of these classes are conjugate to each other. There are six lines through the center of a face (one for each pair of opposite faces), and two rotations of each class per line, so each of these conjugacy classes contains 12 elements.

Finally, all rotations of type (3) are conjugate to each other; there are 15 of them (one line per pair of opposite edges, and one nontrivial rotation per line).

Lastly, the identity forms its own conjugacy class.

Now suppose H is a normal subgroup of D. To be normal, H must be a union of conjugacy classes, and $|H|$ must divide $|D| = 60$. However, we've found conjugacy classes of sizes 1, 12, 12, 15, and 20. There is no way to add up any subset of these numbers to get a divisor of 60, other than 1 (i.e. the trivial subgroup) or 60 (D itself). Thus no nontrivial proper subgroup of D can be a union of conjugacy classes, so D is simple.

Claim 3.4. *The map ϕ_D is injective.*

Proof: The kernel of ϕ_D must be a normal subgroup of D. It cannot be all of D, because D clearly does not act trivially on the set of five cubes. But since D is simple, the only other normal subgroup of D is the trivial one, so ϕ_D must have trivial kernel.

Now we've established that D is isomorphic (via ϕ_D) to an order-60 (index-2) subgroup of S_5. By claim 1.3 above, this implies that $D \simeq A_5$.