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Summary	In these notes we construct a quantization functor, associating a Hilbert space $H(V)$ to a finite dimensional symplectic vector space V over a finite field Fq. As a result, we obtain a canonical model for theWeil representation of the symplectic group $Sp(V)$. The main technical result is a proof of a stronger form of the Stone–von Neumann theorem for the Heisenberg group over Fq. Our result answers, for the case of the Heisenberg group, a question of Kazhdan about the possible existence of a canonical Hilbert space attached to a coadjoint orbit of a general unipotent group over Fq.

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Notes on Canonical Quantization of Symplectic Vector Spaces over Finite Fields

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Summary. In these notes we construct a quantization functor, associating a Hilbert space $\mathcal{H}(V)$ to a finite dimensional symplectic vector space V over a finite field $\mathbb{F}_q$. As a result, we obtain a canonical model for the Weil representation of the symplectic group $Sp(V)$. The main technical result is a proof of a stronger form of the Stone–von Neumann theorem for the Heisenberg group over $\mathbb{F}_q$. Our result answers, for the case of the Heisenberg group, a question of Kazhdan about the possible existence of a canonical Hilbert space attached to a coadjoint orbit of a general unipotent group over $\mathbb{F}_q$.

Key words: Quantization functor, Weil representation, Quantization of Lagrangian correspondences, Geometric intertwining operators.

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1 Introduction

Quantization is a fundamental procedure in mathematics and in physics. Although it is widely used in both contexts, its precise nature remains to some extent unclear. From the physical side, quantization is the procedure by which one associates to a classical mechanical system its quantum counterpart. From the mathematical side, it seems that quantization is a way to construct interesting Hilbert spaces out of symplectic manifolds, suggesting a method for constructing representations of the corresponding groups of symplectomorphisms [14, 16].

Probably, one of the principal manifestation of quantization in mathematics appears in the form of the Weil representation [19, 20, 22] of the metaplectic group

$$\rho : Mp\left(2n, \mathbb{R}\right) \to U\left(L^2\left(\mathbb{R}^n\right)\right),$$

where $Mp(2n, \mathbb{R})$ is a double cover of the linear symplectic group $Sp(2n, \mathbb{R})$. The general ideology [23, 24] suggests that the Weil representation appears through a quantization of the standard symplectic vector space $(\mathbb{R}^{2n}, \omega)$. This means that there should exist a quantization functor $\mathcal{H}$, associating to a symplectic manifold (M, ω) an Hilbert space $\mathcal{H}(M)$, such that when applied to $(\mathbb{R}^{2n}, \omega)$ it yields the Weil representation in the form of

$$\mathcal{H} : Sp(2n, \mathbb{R}) \to U\left(\mathcal{H}\left(\mathbb{R}^{2n}, \omega\right)\right).$$

As stated, this ideology is too naive since it does not account for the metaplectic cover.

1.1 Main results

In these notes, we show that the quantization ideology can be made precise when applied in the setting of symplectic vector spaces over the finite field $\mathbb{F}_q$, where q is odd. Specifically, we construct a quantization functor $\mathcal{H} : \mathsf{Symp} \to \mathsf{Hilb}$, where Symp denotes the (groupoid) category whose objects are finite dimensional symplectic vector spaces over $\mathbb{F}_q$ and morphisms are linear isomorphisms of symplectic vector spaces and Hilb denotes the category of finite dimensional Hilbert spaces.

As a consequence, for a fixed symplectic vector space $V \in \mathsf{Symp}$, we obtain, by functoriality, a homomorphism $\mathcal{H} : Sp(V) \to U(\mathcal{H}(V))$, which we refer to as the canonical model of the Weil representation of the symplectic group $Sp(V)$.

Properties of the quantization functor

In addition, we show that the functor $\mathcal{H}$ satisfies the following basic properties (cf. [24]):

- **Compatibility with Cartesian products.** The functor $\mathcal{H}$ is a monoidal functor: Given $V_1, V_2 \in \mathsf{Symp}$, we have a natural isomorphism

$$\mathcal{H}(V_1 \times V_2) \simeq \mathcal{H}(V_1) \otimes \mathcal{H}(V_2).$$

- **Compatibility with duality.** Given $V = (V, \omega) \in \mathsf{Symp}$, its symplectic dual is $\overline{V} = (V, -\omega)$. There exists a natural non-degenerate pairing

$$\langle \cdot, \cdot \rangle_V : \mathcal{H}\left(\overline{V}\right) \times \mathcal{H}(V) \to \mathbb{C}.$$

- **Compatibility with linear symplectic reduction**. Given $V \in \mathsf{Symp}$, $I \subset V$ an isotropic subspace in V and $o_I \in \wedge^{\text{top}} I$ a non-zero vector, there exists a natural isomorphism

$$\mathcal{H}(V)^I \simeq \mathcal{H}\left(I^\perp / I\right), \tag{1}$$

where $\mathcal{H}(V)^{I}$ stands for the subspace of I-invariant vectors in $\mathcal{H}(V)$ (an operation which will be made precise in the sequel) and $I^{\perp}/I \in \mathsf{Symp}$ is the symplectic reduction of V with respect to I [3]. (A pair (I, o_I), where $o_I \in \wedge^{\mathrm{top}} I$ is non-zero vector, is called an *oriented isotropic subspace*).

Quantization of oriented Lagrangian subspaces. A particular situation is when $I = L$ is a Lagrangian subspace. In this situation, $L^{\perp}/L = 0$ and (1) yields an isomorphism $\mathcal{H}(V)^{L} \simeq \mathcal{H}(0) = \mathbb{C}$, which associates to $1 \in \mathbb{C}$ a vector $v_{L^{\circ}} \in \mathcal{H}(V)$. This means that we establish a mechanism which associates to every oriented Lagrangian subspace in V a well defined vector in $\mathcal{H}(V)$

$$L^{\circ} \longmapsto v_{L^{\circ}} \in \mathcal{H}(V).$$

Interestingly, to the best of our knowledge (cf. [9]), this kind of structure, which exists in the setting of the Weil representation of the group $Sp(V)$, was not observed before.

Quantization of oriented Lagrangian correspondences. It is also interesting to consider simultaneously the compatibility of $\mathcal{H}$ with Cartesian product, duality, and linear symplectic reduction. The first and second properties imply that $\mathcal{H}\left(\overline{V}_1 \times V_2\right)$ is naturally isomorphic to the vector space $\mathrm{Hom}\left(\mathcal{H}(V_1), \mathcal{H}(V_2)\right)$. The third property implies that every oriented Lagrangian L° in $\overline{V}_1 \times V_2$ (i.e., oriented canonical relation from V_1 to V_2 (cf. [23, 24])) can be quantized into a well defined operator

$$L^{\circ} \longmapsto A_{L^{\circ}} \in \mathrm{Hom}\left(\mathcal{H}(V_1), \mathcal{H}(V_2)\right).$$

In this regard, a particular kind of oriented Lagrangian in $\overline{V} \times V$ is the graph Γ_g of a symplectic linear map $g : V \to V$, $g \in Sp(V)$. The orientation is automatic in this case—it is induced from $\omega^{\wedge -n}$, $\dim(V) = 2n$, through the isomorphism $p_V : \Gamma_g \to V$, where $p_V : \overline{V} \times V \to V$ is the projection on the V-coordinate.

A further and more detailed study of these properties will appear in a subsequent work.

The strong Stone–von Neumann theorem

The main technical result of these notes is a proof ([10, 11] unpublished) of a stronger form of the Stone–von Neumann theorem for the Heisenberg group over $\mathbb{F}_q$. In this regard we describe an algebro-geometric object (an ℓ-adic perverse Weil sheaf $\mathcal{K}$), which, in particular, implies the strong Stone–von Neumann theorem. The construction of the sheaf $\mathcal{K}$ is one of the main contributions of this work.

Finally, we note that our result answers, for the case of the Heisenberg group, a question of Kazhdan [13] about the possible existence of a canonical Hilbert space attached to a coadjoint orbit of a general unipotent group over $\mathbb{F}_q$.

We devote the rest of the introduction to an intuitive explanation of the main ideas and results of these notes.

1.2 Quantization of symplectic vectors spaces over finite fields

Let (V, ω) be a $2n$-dimensional symplectic vector space over the finite field $\mathbb{F}_q$, assuming q is odd. The vector space V considered as an abelian group admits a non-trivial central extension H, called the *Heisenberg group*, which can be presented as $H = V \times \mathbb{F}_q$ with center $Z = Z(H) = \{(0, z) : z \in \mathbb{F}_q\}$. The group $Sp = Sp(V)$ acts on H by group automorphisms via its tautological action on the V-coordinate.

The celebrated Stone–von Neumann theorem [18, 21] asserts that given a non-trivial central character $\psi : Z \to \mathbb{C}^{\times}$, there exists a unique (up to isomorphism) irreducible representation $\pi : H \to GL(\mathcal{H})$ such that the center acts by ψ, i.e., $\pi_{|Z} = \psi \cdot \mathrm{Id}_{\mathcal{H}}$. The representation π is called the *Heisenberg representation.*

Choosing a Lagrangian subvector space $L \in \mathrm{Lag}(V)$ (the set $\mathrm{Lag}(V)$ is called the *Lagrangian Grassmanian*) we can define a model $(\pi_L, H, \mathcal{H}_L)$ of the Heisenberg representation, where $\mathcal{H}_L$ consists of functions $f : H \to \mathbb{C}$ satisfying $f(z \cdot l \cdot h) = \psi(z) f(h)$ for every $l \in L$, $z \in Z$ and the action π_L is given by right translation. The problematic issue in this construction is that there is no preferred choice of a Lagrangian subspace $L \in V$ and consequently none of the spaces $\mathcal{H}_L$ admit an action of the group Sp. In fact, an element $g \in Sp$ induces an isomorphism

$$g : \mathcal{H}_L \widetilde{\to} \mathcal{H}_{gL}, \tag{2}$$

for every $L \in \mathrm{Lag}(V)$.

The strong Stone–von Neumann theorem

The strategy that we will employ is: "If you can not choose a preferred Lagrangian subspace then work with all of them simultaneously".

We can think of the system of models $\{\mathcal{H}_L\}$ as a vector bundle $\mathfrak{H}$ on Lag with fibers $\mathfrak{H}_{|L} = \mathcal{H}_L$, the condition (2) means that $\mathfrak{H}$ is equipped with an Sp-equivariant structure and what we seek is a canonical trivialization of $\mathfrak{H}$. More formally, we seek for a canonical system of intertwining morphisms $F_{M,L} \in \mathrm{Hom}_H(\mathcal{H}_L, \mathcal{H}_M)$, for every $L, M \in \mathrm{Lag}(V)$. The existence of such a system is the content of the strong Stone–von Neumann theorem.

Theorem 1 (Strong Stone–von Neumann theorem). *There exists a canonical system of intertwining morphisms* $\{F_{M,L} \in \mathrm{Hom}_H(\mathcal{H}_L, \mathcal{H}_M)\}$ *satisfying the multiplicativity property* $F_{N,M} \circ F_{M,L} = F_{N,L}$, *for every* $N, M, L \in$ *Lag*(V).

Remark 1 (Important remark). It is important to note here that the precise statement involves the finer notion of an oriented Lagrangian subspace [1,17], but for the sake of the introduction we will ignore this technical nuance.

The Hilbert space $\mathcal{H}(V)$ consists of systems of vectors $(v_L \in \mathcal{H}_L)_{L \in \mathrm{Lag}}$ such that $F_{M,L}(v_L) = v_M$, for every $L, M \in \mathrm{Lag}(V)$. The vector space $\mathcal{H}(V)$ can be thought of as the space of horizontal sections of $\mathfrak{H}$.

As it turns out, the symplectic group Sp naturally acts on $\mathcal{H}(V)$. We denote this representation by $(\rho_V, Sp, \mathcal{H}(V))$, and refer to it as the *canonical model* of the Weil representation. We proceed to explain the main underlying idea behind the construction of the system $\{F_{M,L}\}$.

1.3 Canonical system of intertwining morphisms

The construction will be close in spirit to the procedure of "analytic continuation". We consider the subset $U \subset \mathrm{Lag}(V)^2$, consisting of pairs $(L, M) \in \mathrm{Lag}(V)^2$ which are in general position, that is $L \cap M = 0$. The basic idea is that for a pair $(L, M) \in U$, $F_{M,L}$ can be given by an explicit formula—**ansatz**. The main statement is that this formula admits a unique multiplicative extension to the set of all pairs. The extension is constructed using algebraic geometry.

Extension to singular pairs

It will be convenient to work in the setting of *kernels.* In more detail, every intertwining morphism $F \in \mathrm{Hom}_H(\mathcal{H}_L, \mathcal{H}_M)$ can be presented by a kernel function $K \in \mathbb{C}(H, \psi)$ satisfying $K(m \cdot h \cdot l) = K(h)$, for every $m \in M$ and $l \in L$ (we denote by $\mathbb{C}(H, \psi)$ the subspace of functions $f \in \mathbb{C}(H)$ which are ψ-equivariant with respect to the center, that is $f(z \cdot h) = \psi(z) f(h)$, for every $z \in Z$). Moreover, this presentation is unique when $(M, L) \in U$; hence, in this case, we have a unique kernel $K_{M,L}$ representing our given $F_{M,L}$. If we denote by O the set $U \times H$, we see that the collection $\{K_{M,L} : (M, L) \in U\}$ forms a function $K_O \in \mathbb{C}(O)$ given by $K_O(M, L) = K_{M,L}$ for every $(M, L) \in U$. The problem is how to (correctly) extend the function K_O to the set $X = \mathrm{Lag}(V)^2 \times H$. In order to do that, we invoke the procedure of geometrization, which we briefly explain below.

Geometrization

A general ideology due to Grothendieck is that any meaningful set-theoretic object is governed by a more fundamental algebro-geometric one. The procedure by which one translates from the set theoretic setting to algebraic geometry is called *geometrization*, which is a formal procedure by which sets are replaced by algebraic varieties and functions are replaced by certain sheaf-theoretic objects.

The precise setting consists of a set $X = \mathbf{X}(\mathbb{F}_q)$ of rational points of an algebraic variety $\mathbf{X}$, defined over $\mathbb{F}_q$ and a complex valued function $f \in \mathbb{C}(X)$ governed by an ℓ-adic Weil sheaf $\mathcal{F}$.

The variety $\mathbf{X}$ is a space equipped with an automorphism $Fr : \mathbf{X} \to \mathbf{X}$ (called Frobenius), such that the set X is naturally identified with the set of fixed points $X = \mathbf{X}^{Fr}$.

The sheaf $\mathcal{F}$ can be considered as a vector bundle on the variety $\mathbf{X}$, equipped with an endomorphism $\theta : \mathcal{F} \to \mathcal{F}$ which lifts Fr.

The procedure by which f is obtained from $\mathcal{F}$ is called Grothendieck's *sheaf-to-function correspondence* and it can be described, roughly, as follows. Given a point $x \in X$, the endomorphism θ restricts to an endomorphism $\theta_x : \mathcal{F}_{|x} \to \mathcal{F}_{|x}$ of the fiber $\mathcal{F}_{|x}$. The value of f on the point x is defined to be

$$f(x) = \mathrm{Tr}(\theta_x : \mathcal{F}_{|x} \to \mathcal{F}_{|x}).$$

The function defined by this procedure is denoted by $f = f^{\mathcal{F}}$.

Solution to the extension problem

Our extension problem fits nicely into the geometrization setting: The sets O, X are sets of rational points of corresponding algebraic varieties $\mathbf{O}, \mathbf{X}$, the imbedding $j : O \hookrightarrow X$ is induced from an open imbedding $j : \mathbf{O} \hookrightarrow \mathbf{X}$ and, finally, the function K_O comes from a Weil sheaf $\mathcal{K}_{\mathbf{O}}$ on the variety $\mathbf{O}$.

The extension problem is solved as follows: First extend the sheaf $\mathcal{K}_{\mathbf{O}}$ to a sheaf $\mathcal{K}$ on the variety $\mathbf{X}$ and then take the corresponding function $K = f^{\mathcal{K}}$, which establishes the desired extension. The reasoning behind this strategy is that in the realm of sheaves there exist several functorial operations of extension, probably the most interesting one is called *perverse extension* [2]. The sheaf $\mathcal{K}$ is defined as the perverse extension of $\mathcal{K}_{\mathbf{O}}$.

1.4 Structure of the notes

Apart from the introduction, the notes consists of three sections.

In Section 2, all basic constructions are introduced and main statements are formulated. We begin with the definition of the Heisenberg group and the Heisenberg representation. Next, we introduce the canonical system of intertwining morphisms between different models of the Heisenberg representation and formulate the strong Stone von–Neumann theorem (Theorem 3). We proceed to explain how to present an intertwining morphism by a kernel function, and we reformulate the strong Stone von–Neumann theorem in the setting of kernels (Theorem 4). Using Theorem 3, we construct a quantization functor $\mathcal{H}$. We finish this section by showing that $\mathcal{H}$ is a monoidal functor and that it is compatible with duality and the operation of linear symplectic reduction. In section 3, we construct a sheaf theoretic counterpart for the canonical system of intertwining morphisms (Theorem 5). This sheaf is then used to prove Theorem 4. Finally, in Section 4 we sketch the proof of Theorem 5. Complete proofs for the statements appearing in these notes will appear elsewhere.

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2 Quantization of symplectic vector spaces over finite fields

2.1 The Heisenberg group

Let (V, ω) be a $2n$-dimensional symplectic vector space over the finite field $\mathbb{F}_q$. Considering V as an abelian group, it admits a non-trivial central extension called the *Heisenberg* group. Concretely, the group $H = H(V)$ can be presented as the set $H = V \times \mathbb{F}_q$ with the multiplication given by

$$(v, z) \cdot (v', z') = \Big(v + v', z + z' + \frac{1}{2}\omega(v, v')\Big).$$

The center of H is $Z = Z(H) = \{(0, z) : z \in \mathbb{F}_q\}$. The symplectic group $Sp = Sp(V)$ acts by automorphism of H through its tautological action on the V-coordinate.

2.2 The Heisenberg representation

One of the most important attributes of the group H is that it admits, principally, a unique irreducible representation. We will call this property *The Stone–von Neumann property* (S-vN for short). The precise statement goes as follows. Let $\psi : Z \to \mathbb{C}^\times$ be a non-trivial character of the center. For example we can take $\psi(z) = e^{\frac{2\pi i}{p}\mathrm{tr}(z)}$. It is not hard to show

Theorem 2 (Stone–von Neumann property). *There exists a unique (up to isomorphism) irreducible unitary representation $(\pi, H, \mathcal{H})$ with the center acting by ψ, i.e., $\pi_{|Z} = \psi \cdot Id_{\mathcal{H}}$.*

The representation π which appears in the above theorem will be called the *Heisenberg representation.*

2.3 The strong Stone–von Neumann property

Although the representation π is unique, it admits a multitude of different models (realizations); in fact this is one of its most interesting and powerful attributes. These models appear in families. In this work we will be interested in a particular family of such models which are associated with Lagrangian subspaces in V.

Let us denote by $\mathrm{Lag} = \mathrm{Lag}(V)$ the set of Lagrangian subspaces in V. Let $\mathbb{C}(H,\psi)$ denote the subspace of functions $f \in \mathbb{C}(H)$, satisfying the equivariance property $f(z \cdot h) = \psi(z) f(h)$, for every $z \in Z$.

Given a Lagrangian subspace $L \in \mathrm{Lag}$, we can construct a model $(\pi_L, H, \mathcal{H}_L)$ of the Heisenberg representation: The vector space $\mathcal{H}_L$ consists of functions $f \in \mathbb{C}(H,\psi)$ satisfying $f(l \cdot h) = f(h)$, for every $l \in L$ and the Heisenberg action is given by right translation $(\pi_L(h) \triangleright f)(h') = f(h' \cdot h)$, for $f \in \mathcal{H}_L$.

Definition 1. *An oriented Lagrangian L° is a pair $L^\circ = (L, o_L)$, where L is a Lagrangian subspace in V and o_L is a non-zero vector in $\bigwedge^{top} L$.*

Let $\mathrm{Lag}^\circ = \mathrm{Lag}^\circ(V)$ denote the set of oriented Lagrangian subspaces in V. We associate to each oriented Lagrangian subspace L°, a model $(\pi_{L^\circ}, H, \mathcal{H}_{L^\circ})$ of the Heisenberg representation simply by forgetting the orientation, taking $\mathcal{H}_{L^\circ} = \mathcal{H}_L$ and $\pi_{L^\circ} = \pi_L$. Sometimes, we will use a more informative notation $\mathcal{H}_{L^\circ} = \mathcal{H}_{L^\circ}(V)$ or $\mathcal{H}_{L^\circ} = \mathcal{H}_{L^\circ}(V,\psi)$.

Canonical system of intertwining morphisms

Given a pair $(M^\circ, L^\circ) \in \mathrm{Lag}^{\circ 2}$, the models $\mathcal{H}_{L^\circ}$ and $\mathcal{H}_{M^\circ}$ are isomorphic as representations of H by Theorem 2, moreover, since the Heisenberg representation is irreducible, the vector space $\mathrm{Hom}_H(\mathcal{H}_{L^\circ}, \mathcal{H}^\circ_M)$ of intertwining morphisms is one-dimensional. Roughly, the strong Stone–von Neumann property asserts the existence of a distinguished element $F_{M^\circ,L^\circ} \in \mathrm{Hom}_H(\mathcal{H}_{L^\circ}, \mathcal{H}^\circ_M)$, for every pair $(M^\circ, L^\circ) \in \mathrm{Lag}^{\circ 2}$. The precise statement involves the following definition:

Definition 2. *A system $\{F_{M^\circ,L^\circ} \in \mathrm{Hom}_H(\mathcal{H}_{L^\circ}, \mathcal{H}^\circ_M) : (M^\circ, L^\circ) \in Lag^{\circ 2}\}$ of intertwining morphisms is called multiplicative if for every triple $(N^\circ, M^\circ, L^\circ) \in Lag^{\circ 3}$ the following equation holds*

$$F_{N^\circ,L^\circ} = F_{N^\circ,M^\circ} \circ F_{M^\circ,L^\circ}.$$

We proceed as follows. Let $U \subset \mathrm{Lag}^{\circ 2}$ denote the set of pairs $(M^\circ, L^\circ) \in \mathrm{Lag}^{\circ 2}$ which are in general position, i.e., $L \cap M = 0$. For $(M^\circ, L^\circ) \in U$, we define F_{M°,L° by the following explicit formula:

$$F_{M^\circ,L^\circ} = C_{M^\circ,L^\circ} \cdot \widetilde{F}_{M,L}, \tag{3}$$

where $\widetilde{F}_{M,L} : \mathcal{H}_{L^\circ} \to \mathcal{H}_{M^\circ}$ is the averaging morphism

$$\widetilde{F}_{M,L}[f](h) = \sum_{m \in M} f(m \cdot h),$$

for every $f \in \mathcal{H}_{L^\circ}$ and C_{M°,L° is a normalization constant given by

$$C_{M^\circ,L^\circ} = (G_1/q)^n \cdot \sigma\left((-1)^{\binom{n}{2}} \omega_\wedge (o_L, o_M)\right),$$

where $n = \frac{\dim(V)}{2}$, σ is the unique quadratic character (also called the Legendre character) of the multiplicative group $G_m = \mathbb{F}_q^\times$, G_1 is the one-dimensional Gauss sum

$$G_1 = \sum_{z \in \mathbb{F}_q} \psi\left(\frac{1}{2} z^2\right),$$

and $\omega_\wedge$ is the pairing $\omega_\wedge : \bigwedge^{\mathrm{top}} L \bigotimes \bigwedge^{\mathrm{top}} M \to \mathbb{F}_q$ induced by the symplectic form.

Theorem 3 (The strong Stone–von Neumann property). *There exists a unique system $\{F_{M^\circ,L^\circ}\}$ of intertwining morphisms satisfying*

1. *Restriction. For every pair $(M^\circ, L^\circ) \in U$, F_{M°,L° is given by (3).*
2. *Multiplicativity. For every triple $(N^\circ, M^\circ, L^\circ) \in Lag^{\circ 3}$,*

$$F_{N^\circ,L^\circ} = F_{N^\circ,M^\circ} \circ F_{M^\circ,L^\circ}.$$

Theorem 3 will follow from Theorem 4 below.

Granting the existence and uniqueness of the system $\{F_{M^\circ,L^\circ}\}$, we can write F_{M°,L° in a closed form, for a general pair $(M^\circ, L^\circ) \in \mathrm{Lag}^{\circ 2}$. In order to do that we need to fix some additional terminology.

Let $I = M \cap L$. We have canonical tensor product decompositions

$$\bigwedge^{\mathrm{top}} M = \bigwedge^{\mathrm{top}} I \bigotimes \bigwedge^{\mathrm{top}} M/I,$$
$$\bigwedge^{\mathrm{top}} L = \bigwedge^{\mathrm{top}} I \bigotimes \bigwedge^{\mathrm{top}} L/I.$$

In terms of the above decompositions, the orientation can be written in the form $o_M = \iota_M \otimes o_{M/I}$, $o_L = \iota_L \otimes o_{L/I}$. Using the same notations as before, we denote by $\widetilde{F}_{M,L} : \mathcal{H}_{L^\circ} \to \mathcal{H}_{M^\circ}$ the averaging morphism

$$\widetilde{F}_{M,L}[f](h) = \sum_{\overline{m} \in M/I} f(m \cdot h),$$

for $f \in \mathcal{H}_{L^\circ}$ and by C_{M°,L° the normalization constant

$$C_{M^\circ,L^\circ} = (G_1)^k \cdot \sigma\left((-1)^{\binom{k}{2}} \frac{\iota_M}{\iota_L} \cdot \omega_\wedge\left(o_{L/I}, o_{M/I}\right)\right),$$

where $k = \dfrac{\dim(I^\perp/I)}{2}$.

Proposition 1. *For every* $(M^{\circ}, L^{\circ}) \in Lag^{\circ 2}$

$$F_{M^{\circ},L^{\circ}} = C_{M^{\circ},L^{\circ}} \cdot \widetilde{F}_{M,L}.$$

2.4 Kernel presentation of an intertwining morphism

An explicit way to present an intertwining morphism is via a kernel function. Fix a pair $(M^{\circ}, L^{\circ}) \in \mathrm{Lag}^{\circ 2}$ and let $\mathbb{C}\,(M\backslash H/L, \psi)$ denote the subspace of functions $f \in \mathbb{C}\,(H, \psi)$ satisfying the equivariance property $f\,(m \cdot h \cdot l) = f\,(h)$ for every $m \in M$ and $l \in L$. Given a function $K \in \mathbb{C}\,(M\backslash H/L, \psi)$, we can associate to it an intertwining morphism $I\,[K] \in \mathrm{Hom}_H(\mathcal{H}_{L^{\circ}}, \mathcal{H}_M^{\circ})$ defined by

$$I\,[K]\,(f) = K * f = m_!\,(K \boxtimes_{Z \cdot M} f)\,,$$

for every $f \in \mathcal{H}_{L^{\circ}}$. Here, $K \boxtimes_{Z \cdot L} f$ denotes the function $K \boxtimes f \in \mathbb{C}\,(H \times H)$, factored to the quotient $H \times_{Z \cdot L} H$ and $m_!$ denotes the operation of summation along the fibers of the multiplication mapping $m : H \times H \to H$. The function K is called an *intertwining kernel*. The procedure just described defines a linear transform

$$I : \mathbb{C}\,(M\backslash H/L, \psi) \longrightarrow \mathrm{Hom}_H(\mathcal{H}_{L^{\circ}}, \mathcal{H}_{M^{\circ}}).$$

An easy verification reveals that I is surjective, but it is injective only when M, L are in general position.

Fix a triple $(N^{\circ}, M^{\circ}, L^{\circ}) \in \mathrm{Lag}^{\circ 3}$. Given kernels $K_1 \in \mathbb{C}\,(N\backslash H/M, \psi)$ and $K_2 \in \mathbb{C}\,(M\backslash H/L, \psi)$, their convolution $K_1 * K_2 = m_!\,(K_1 \boxtimes_{Z \cdot M} K_2)$ lies in $\mathbb{C}\,(N\backslash H/L, \psi)$. The transform I sends convolution of kernels to composition of operators

$$I\,[K_1 * K_2] = I\,[K_1] \circ I\,[K_2]\,.$$

Canonical system of intertwining kernels

Below, we formulate a slightly stronger version of Theorem 3, in the setting of kernels.

Definition 3. *A system* $\{K_{M^{\circ},L^{\circ}} \in \mathbb{C}\,(M\backslash H/L, \psi) : (M^{\circ}, L^{\circ}) \in Lag^{\circ 2}\}$ *of kernels is called* <u>*multiplicative*</u> *if for every triple* $(N^{\circ}, M^{\circ}, L^{\circ}) \in Lag^{\circ 3}$ *the following equation holds*

$$K_{N^{\circ},L^{\circ}} = K_{N^{\circ},M^{\circ}} * K_{M^{\circ},L^{\circ}}$$

A multiplicative system of kernels $\{K_{M^{\circ},L^{\circ}}\}$ can be equivalently thought of as a single function $K \in \mathbb{C}\left(\mathrm{Lag}^{\circ 2} \times H\right)$, $K\,(M^{\circ}, L^{\circ}) = K_{M^{\circ},L^{\circ}}$, satisfying the following multiplicativity relation on $\mathrm{Lag}^{\circ 3} \times H$

$$p_{12}^{*}K * p_{23}^{*}K = p_{13}^{*}K, \tag{4}$$

where $p_{ij}\left(\left(L_1^\circ, L_2^\circ, L_3^\circ\right), h\right) = \left(\left(L_i^\circ, L_j^\circ\right), h\right)$ are the projections on the i, j copies of Lag° and the left-hand side of (4) means fiberwise convolution, namely $p_{12}^* K * p_{23}^* K(L_1^\circ, L_2^\circ, L_3^\circ) = K\left(L_1^\circ, L_2^\circ\right) * K\left(L_2^\circ, L_3^\circ\right)$. To simplify notations, we will sometimes suppress the projections p_{ij} from (4) obtaining a much cleaner formula

$$K * K = K.$$

We proceed along lines similar to Section 2.3. For every $(M^\circ, L^\circ) \in U$, there exists a unique kernel $K_{M^\circ,L^\circ} \in \mathbb{C}\left(M\backslash H/L, \psi\right)$ such that $F_{M^\circ,L^\circ} = I\left[K_{M^\circ,L^\circ}\right]$, which is given by the following explicit formula

$$K_{M^\circ,L^\circ} = C_{M^\circ,L^\circ} \cdot \widetilde{K}_{M^\circ,L^\circ}, \tag{5}$$

where $\widetilde{K}_{M^\circ,L^\circ} = \left(\iota^{-1}\right)^* \psi$, $\iota = \iota_{M^\circ,L^\circ}$ is the isomorphism given by the composition $Z \hookrightarrow H \twoheadrightarrow M\backslash H/L$. The system $\{K_{M^\circ,L^\circ} : (M^\circ, L^\circ) \in U\}$ yields a well defined function $K_U \in \mathbb{C}\left(U \times H\right)$.

Theorem 4 (Canonical system of kernels). *There exists a unique function $K \in \mathbb{C}\left(Lag^{\circ 2} \times H\right)$ satisfying*

1. *Restriction.* $K_{|U} = K_U$.
2. *Multiplicativity.* $K * K = K$.

We note that the proof of the uniqueness part in Theorem 4 is easy, it follows from the fact that for every pair $N^\circ, L^\circ \in \mathrm{Lag}^\circ$ one can find a third $M^\circ \in \mathrm{Lag}^\circ$ such that the pairs N°, M° and M°, L° are in general position. Therefore, by the multiplicativity property (Property 2), $K_{N^\circ,L^\circ} = K_{N^\circ,M^\circ} * K_{M^\circ,L^\circ}$. The proof of the existence part will be algebro-geometric (see Section 3). Finally, we note that Theorem 3 follows from Theorem 4 by taking $F_{M^\circ,L^\circ} = I\left[K_{M^\circ,L^\circ}\right]$.

2.5 The canonical vector space

Let us denote by Symp the category whose objects are symplectic vector spaces over $\mathbb{F}_q$ and morphisms are linear isomorphisms of symplectic vector spaces. Using the canonical system of intertwining morphisms $\{F_{M^\circ,L^\circ}\}$ we can associate, in a functorial manner, a vector space $\mathcal{H}(V)$ to a symplectic vector space $V \in$ Symp. The construction proceeds as follows.

Let $\mathit{\Gamma}(V)$ denote the total vector space

$$\mathit{\Gamma}(V) = \bigoplus_{L^\circ \in \mathrm{Lag}^\circ(V)} \mathcal{H}_{L^\circ},$$

Define $\mathcal{H}(V)$ to be the subvector space of $\mathit{\Gamma}(V)$ consisting of sequences $(v_{L^\circ} \in \mathcal{H}_{L^\circ} : L^\circ \in \mathrm{Lag}^\circ)$ satisfying $F_{M^\circ,L^\circ}(v_{L^\circ}) = v_{M^\circ}$ for every $(M^\circ, L^\circ) \in \mathrm{Lag}^{\circ 2}(V)$. We will call the vector space $\mathcal{H}(V)$ the *canonical vector space* associated with V. Sometimes we will use the more informative notation $\mathcal{H}(V) = \mathcal{H}(V, \psi)$.

Proposition 2 (Functoriality). *The rule $V \mapsto \mathcal{H}(V)$ establishes a contravariant (quantization) functor*

$$\mathcal{H} : \mathsf{Symp} \longrightarrow \mathsf{Vect},$$

where Vect *denote the category of finite dimensional complex vector spaces.*

Considering a fixed symplectic vector space V, we obtain as a consequence a representation $(\rho_V, Sp(V), \mathcal{H}(V))$, with $\rho_V(g) = \mathcal{H}(g^{-1})$, for every $g \in Sp(V)$. The representation ρ_V is isomorphic to the Weil representation and we call it the *canonical model* of the Weil representation.

Remark 2. The canonical model ρ_V can be viewed from another perspective: We begin with the total vector space Γ and make the following two observations. First observation is that the symplectic group Sp acts naturally on Γ, the action is of a geometric nature, i.e., induced from the diagonal action on $\mathrm{Lag}^{\circ} \times H$. Second observation is that the system $\{F_{M^{\circ},L^{\circ}}\}$ defines an Sp-invariant idempotent (total Fourier transform) $F : \Gamma \to \Gamma$ given by

$$F(v_{L^{\circ}}) = \frac{1}{\#(\mathrm{Lag}^{\circ})} \bigoplus_{M^{\circ} \in \mathrm{Lag}^{\circ}} F_{M^{\circ},L^{\circ}}(v_{L^{\circ}}),$$

for every $L^{\circ} \in \mathrm{Lag}^{\circ}$ and $v_{L^{\circ}} \in \mathcal{H}_{L^{\circ}}$. The situation is summarized in the following diagram:

$$Sp \circlearrowright \Gamma \circlearrowleft F.$$

The canonical model is given by the image of F, that is, $\mathcal{H}(V) = F\Gamma$. The nice thing about this point of view is that it shows a clear distinction between operators associated with action of the symplectic group and operators associated with intertwining morphisms. Finally, we remark that one can also consider the Sp-invariant idempotent $F^{\perp} = \mathrm{Id} - F$ and the associated representation $\left(\rho_V^{\perp}, Sp, \mathcal{H}(V)^{\perp}\right)$, with $\mathcal{H}(V)^{\perp} = F^{\perp}\Gamma$. The meaning of this representation is unclear.

Compatibility with Cartesian products

The category Symp admits a monoidal structure given by Cartesian product of symplectic vector spaces. The category Vect admits the standard monoidal structure given by tensor product. With respect to these monoidal structures, the functor $\mathcal{H}$ is a monoidal functor.

Proposition 3. *There exists a natural isomorphism*

$$\alpha_{V_1 \times V_2} : \mathcal{H}(V_1 \times V_2) \to \mathcal{H}(V_1) \otimes \mathcal{H}(V_2),$$

where $V_1, V_2 \in \mathsf{Symp}$.

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As a result, we obtain the following compatibility condition between the canonical models of the Weil representation

$$\alpha_{V_1 \times V_2} : \left(\rho_{V_1 \times V_2}\right)_{|Sp(V_1) \times Sp(V_2)} \longrightarrow \rho_{V_1} \otimes \rho_{V_2}. \tag{6}$$

Remark 3 ([5]). Condition (6) has an interesting consequence in case the ground field is $\mathbb{F}_3$. In this case, the group $Sp(V)$ is not perfect when $\dim(V) = 2$, therefore, a priori, the Weil representation is not uniquely defined in this particular situation. However, since the group $Sp(V)$ becomes perfect when $\dim(V) > 2$, the canonical model gives a natural choice for the Weil representation in the *singular* dimension, $\dim(V) = 2$.

Compatibility with symplectic duality

Let $V = (V, \omega) \in \mathsf{Symp}$ and let us denote by $\overline{V} = (V, -\omega)$ the symplectic dual of V.

Proposition 4. *There exists a natural non-degenerate pairing*

$$\langle \cdot, \cdot \rangle_V : \mathcal{H}\left(\overline{V}, \psi\right) \times \mathcal{H}(V, \psi) \to \mathbb{C},$$

where $V \in \mathsf{Symp}$.

Compatibility with symplectic reduction

Let $V \in \mathsf{Symp}$ and let I be an isotropic subspace in V considered as an abelian subgroup in $H(V)$. On the one hand, we can associate to I the subspace $\mathcal{H}(V)^I$ of I-invariant vectors. On the other hand, we can form the symplectic reduction $I^{\perp}/I$ and consider the vector space $\mathcal{H}\left(I^{\perp}/I\right)$ (note that since I is isotropic then $I \subset I^{\perp}$ and $I^{\perp}/I$ is equipped with a natural symplectic structure). Roughly, we claim that the vector spaces $\mathcal{H}\left(I^{\perp}/I\right)$ and $\mathcal{H}(V)^I$ are naturally isomorphic. The precise statement involves the following definition.

Definition 4. *An oriented isotropic subspace in V is a pair $I^{\circ} = (I, o_I)$, where $I \subset V$ is an isotropic subspace and o_I is a non-trivial vector in $\bigwedge^{top} I$.*

Proposition 5. *There exists a natural isomorphism*

$$\alpha_{(I^{\circ}, V)} : \mathcal{H}(V)^I \to \mathcal{H}\left(I^{\perp}/I\right),$$

where, $V \in \mathsf{Symp}$ and I° an oriented isotropic subspace in V. The naturality condition is $\mathcal{H}(f_I) \circ \alpha_{(J^{\circ}, U)} = \alpha_{(I^{\circ}, V)} \circ \mathcal{H}(f)$, for every $f \in \mathrm{Mor}_{\mathsf{Symp}}(V, U)$ such that $f(I^{\circ}) = J^{\circ}$ and $f_I \in \mathrm{Mor}_{\mathsf{Symp}}\left(I^{\perp}/I, J^{\perp}/J\right)$ is the induced morphism.

As a result we obtain another compatibility condition between the canonical models of the Weil representation. In order to see this, fix $V \in \mathsf{Symp}$ and let I° be an oriented isotropic subspace in V. Let $P \subset Sp(V)$ be the subgroup of elements $g \in Sp(V)$ such that $g(I^{\circ}) = I^{\circ}$. The isomorphism $\alpha_{(I^{\circ},V)}$ establishes the following isomorphism:

$$\alpha_{(I^{\circ},V)} : (\rho_V)_{|P} \longrightarrow \rho_{I^{\perp}/I} \circ \pi, \tag{7}$$

where $\pi : P \rightarrow Sp\left(I^{\perp}/I\right)$ is the canonical homomorphism.

3 Geometric intertwining morphisms

In this section we are going to prove Theorem 4, by constructing a geometric counterpart to the set-theoretic system of intertwining kernels. This will be achieved using geometrization.

3.1 Preliminaries from algebraic geometry

We denote by k an algebraic closure of $\mathbb{F}_q$. Next we have to take some space to recall notions and notations from algebraic geometry and the theory of ℓ-adic sheaves.

Varieties

In the sequel, we are going to translate back and forth between algebraic varieties defined over the finite field $\mathbb{F}_q$ and their corresponding sets of rational points. In order to prevent confusion between the two, we use bold-face letters to denote a variety $\mathbf{X}$ and normal letters X to denote its corresponding set of rational points $X = \mathbf{X}(\mathbb{F}_q)$. For us, a variety $\mathbf{X}$ over the finite field is a quasi-projective algebraic variety, such that the defining equations are given by homogeneous polynomials with coefficients in the finite field $\mathbb{F}_q$. In this situation, there exists a (geometric) *Frobenius* endomorphism $Fr : \mathbf{X} \rightarrow \mathbf{X}$, which is a morphism of algebraic varieties. We denote by X the set of points fixed by Fr, i.e.,

$$X = \mathbf{X}(\mathbb{F}_q) = \mathbf{X}^{Fr} = \{x \in \mathbf{X} : Fr(x) = x\}.$$

The category of algebraic varieties over $\mathbb{F}_q$ will be denoted by $\mathsf{Var}_{\mathbb{F}_q}$.

Sheaves

Let $\mathsf{D}^b(\mathbf{X})$ denote the bounded derived category of constructible ℓ-adic sheaves on $\mathbf{X}$ [2,4]. We denote by $\mathsf{Perv}(\mathbf{X})$ the Abelian category of perverse sheaves on the variety $\mathbf{X}$, i.e., the heart with respect to the autodual perverse t-structure

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in $\mathsf{D}^b(\mathbf{X})$. An object $\mathcal{F} \in \mathsf{D}^b(\mathbf{X})$ is called n-perverse if $\mathcal{F}[n] \in \mathsf{Perv}(\mathbf{X})$. Finally, we recall the notion of a Weil structure (Frobenius structure) [4]. A Weil structure associated to an object $\mathcal{F} \in \mathsf{D}^b(\mathbf{X})$ is an isomorphism

$$\theta : Fr^*\mathcal{F} \longrightarrow \mathcal{F}.$$

A pair $(\mathcal{F}, \theta)$ is called a Weil object. By an abuse of notation we often denote θ also by Fr. We choose once an identification $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, hence all sheaves are considered over the complex numbers.

Remark 4. All the results in this section make perfect sense over the field $\overline{\mathbb{Q}}_\ell$, in this respect the identification of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ is redundant. The reason it is specified is in order to relate our results with the standard constructions of the Weil representation [7, 12].

Given a Weil object $(\mathcal{F}, Fr^*\mathcal{F} \simeq \mathcal{F})$ one can associate to it a function $f^{\mathcal{F}} : X \to \mathbb{C}$ to $\mathcal{F}$ as follows

$$f^{\mathcal{F}}(x) = \sum_i (-1)^i \mathrm{Tr}(Fr_{|H^i(\mathcal{F}_x)}).$$

This procedure is called *Grothendieck's sheaf-to-function correspondence.* Another common notation for the function $f^{\mathcal{F}}$ is $\chi_{Fr}(\mathcal{F})$, which is called the *Euler characteristic* of the sheaf $\mathcal{F}$.

3.2 Canonical system of geometric intertwining kernels

We shall now start the geometrization procedure.

Replacing sets by varieties

The first step we take is to replace all sets involved by their geometric counterparts, i.e., algebraic varieties. The symplectic space (V, ω) is naturally identified as the set $V = \mathbf{V}(\mathbb{F}_q)$, where $\mathbf{V}$ is a $2n$-dimensional symplectic vector space in $\mathsf{Var}_{\mathbb{F}_q}$. The Heisenberg group H is naturally identified as the set $H = \mathbf{H}(\mathbb{F}_q)$, where $\mathbf{H} = \mathbf{V} \times \mathbb{A}^1$ is the corresponding group variety. Finally, $\mathrm{Lag}^\circ = \mathbf{Lag}^\circ(\mathbb{F}_q)$, where $\mathbf{Lag}^\circ$ is the variety of oriented Lagrangians in $\mathbf{V}$.

Replacing functions by sheaves

The second step is to replace functions by their sheaf-theoretic counterparts [6]. The additive character $\psi : \mathbb{F}_q \longrightarrow \mathbb{C}^\times$ is associated via the sheaf-to-function correspondence to the Artin–Schreier sheaf $\mathcal{L}_\psi$ living on $\mathbb{A}^1$, i.e., we have $f^{\mathcal{L}_\psi} = \psi$. The Legendre character σ on $\mathbb{F}_q^\times \simeq \mathbb{G}_m(\mathbb{F}_q)$ is associated to the Kummer sheaf $\mathcal{L}_\sigma$ on $\mathbb{G}_m$. The one-dimensional Gauss sum G_1 is associated with the Weil object

$$\mathcal{G}_1 = \int_{\mathbb{A}^1} \mathcal{L}_{\psi(z^2)} \in \mathsf{D}^b(\mathbf{pt}),$$

where, for the rest of these notes, $\int = \int_!$ denotes integration with compact support [2]. Grothendieck's Lefschetz trace formula [8] implies that, indeed, $f^{\mathcal{G}_1} = G_1$. In fact, there exists a quasi-isomorphism $\mathcal{G}_1 \longrightarrow H^1(\mathcal{G}_1)[-1]$ and $\dim H^1(\mathcal{G}_1) = 1$, hence, $\mathcal{G}_1$ can be thought of as a one-dimensional vector space, equipped with a Frobenius operator, sitting at cohomological degree 1.

Our main objective, in this section, is to construct a multiplicative system of kernels $K : \mathrm{Lag}^{\circ 2} \times H \longrightarrow \mathbb{C}$ extending the subsystem K_U (see 2.4). The extension appears as a direct consequence of the following geometrization theorem:

Theorem 5 (Geometric kernel sheaf). *There exists a geometrically irreducible* $[\dim(\mathbf{Lag}^{\circ 2}) + \mathbf{n} + \mathbf{1}]$*-perverse Weil sheaf* $\mathcal{K}$ *on* $\mathbf{Lag}^{\circ 2} \times \mathbf{H}$ *of pure weight* $w(\mathcal{K}) = 0$*, satisfying the following properties:*

1. Multiplicativity property. There exists an isomorphism

$$\mathcal{K} \simeq \mathcal{K} * \mathcal{K}.$$

2. Function property. We have $f^{\mathcal{K}}_{|U} = K_U$.

For a proof, see Section 4.

Proof of Theorem 4

Let $K = f^{\mathcal{K}}$. Invoking Theorem 5, we obtain that K is multiplicative (Property 1) and extends K_U (Property 2). Hence, we see that K satisfies the conditions of Theorem 4. The nice thing about this construction is that it uses geometry and, in particular, the notion of perverse extension which has no counterpart in the set-function theoretic setting.

4 Proof of the geometric kernel sheaf theorem

Section 4 is devoted to sketching the proof of Theorem 5.

4.1 Construction

The construction of the sheaf $\mathcal{K}$ is based on formula (5). Let $\mathbf{U} \subset \mathbf{Lag}^{\circ 2}$ be the open subvariety consisting of pairs $(M^\circ, L^\circ) \in \mathbf{Lag}^{\circ 2}$ in general position. The construction proceeds as follows:

- *Non-normalized kernel.* On the variety $\mathbf{U} \times \mathbf{H}$ define the sheaf

$$\widetilde{\mathcal{K}}_{\mathbf{U}}(M^\circ, L^\circ) = \left(\iota^{-1}\right)^* \mathcal{L}_\psi,$$

where $\iota = \iota_{M^\circ, L^\circ}$ is the composition $\mathbf{Z} \hookrightarrow \mathbf{H} \twoheadrightarrow \mathbf{M} \backslash \mathbf{H} / \mathbf{L}$.

- *Normalization coefficient.* On the open subvariety $\mathbf{U} \times \mathbf{H}$ define the sheaf

$$\mathcal{C}\left(M^{\circ}, L^{\circ}\right) = \mathcal{G}_1^{\otimes n} \otimes \mathcal{L}_{\sigma}\left((-1)^{\binom{n}{2}} \omega_{\wedge}\left(o_L, o_M\right)\right)[2n](n). \tag{8}$$

- *Normalized kernels.* On the open subvariety $\mathbf{U} \times \mathbf{H}$ define the sheaf

$$\mathcal{K}_{\mathbf{U}} = \mathcal{C} \otimes \widetilde{\mathcal{K}}_{\mathbf{U}}.$$

Finally, take

$$\mathcal{K} = j_{!*}\mathcal{K}_{\mathbf{U}}, \tag{9}$$

where $j : \mathbf{U} \times \mathbf{H} \hookrightarrow \mathbf{Lag}^{\circ 2} \times \mathbf{H}$ is the open imbedding, and $j_{!*}$ is the functor of perverse extension [2] (in our setting, $j_{!*}$ might better be called irreducible extension, since the sheaves we consider are not perverse but perverse up to a cohomological shift). It follows directly from the construction that the sheaf $\mathcal{K}$ is irreducible $[\dim(\mathbf{Lag}^{\circ 2}) + n + 1]$-perverse of pure weight 0.

The function property (Property 2) is clear from the construction. We are left to prove the multiplicativity property (Property 2).

4.2 Proof of the multiplicativity property

We need to show that

$$p_{13}^{*}\mathcal{K} \simeq p_{12}^{*}\mathcal{K} * p_{23}^{*}\mathcal{K}, \tag{10}$$

where $p_{ij} : \mathbf{Lag}^{\circ 3} \times \mathbf{H} \to \mathbf{Lag}^{\circ 2} \times \mathbf{H}$ are the projectors on the i, j copies of $\mathbf{Lag}^{\circ}$. We will need the following notations. Let $\mathbf{U}_3 \subset \mathbf{Lag}^{\circ 3}$ denote the open subvariety consisting of triples $(L_1^{\circ}, L_2^{\circ}, L_3^{\circ})$ which are in general position pairwisely. Let $n_k = \dim(\mathbf{Lag}^{\circ k}) + n + 1$.

Lemma 1. *There exists, on* $\mathbf{U}_3 \times \mathbf{H}$*, an isomorphism*

$$p_{13}^{*}\mathcal{K} \simeq p_{12}^{*}\mathcal{K} * p_{23}^{*}\mathcal{K}.$$

Let $\mathbf{V}_3 \subset \mathbf{Lag}^{\circ 2}$ be the open subvariety consisting of triples $(L_1^{\circ}, L_2^{\circ}, L_3^{\circ}) \in \mathbf{Lag}^{\circ 4}$ such that L_1°, L_2° and L_2°, L_3° are in general position. Lemma 1 admits a slightly stronger form.

Lemma 2. *There exists, on* $\mathbf{V}_3 \times \mathbf{H}$*, an isomorphism*

$$p_{13}^{*}\mathcal{K} \simeq p_{12}^{*}\mathcal{K} * p_{23}^{*}\mathcal{K}.$$

We can now finish the proof of (10). Lemma 1 implies that the sheaves $p_{13}^{*}\mathcal{K}$ and $p_{12}^{*}\mathcal{K} * p_{23}^{*}\mathcal{K}$ are isomorphic on the open subvariety $\mathbf{U}_3 \times \mathbf{H}$. The sheaf $p_{13}^{*}\mathcal{K}$ is irreducible $[n_3]$-perverse as a pullback by a smooth, surjective with connected fibers morphism, of an irreducible $[n_2]$-perverse sheaf on $\mathbf{Lag}^{\circ 2} \times \mathbf{H}$. Hence, it is enough to show that the sheaf $p_{12}^{*}\mathcal{K} * p_{23}^{*}\mathcal{K}$ irreducible $[n_3]$-perverse. Let $\mathbf{V}_4 \subset \mathbf{Lag}^{\circ 4}$ be the open subvariety consisting of quadruples $(L_1^{\circ}, L_2^{\circ}, L_3^{\circ}, L_4^{\circ}) \in \mathbf{Lag}^{\circ 4}$ such that the pairs L_1°, L_2° and L_2°, L_3° are in general position. Consider the projection $p_{134} : \mathbf{V}_4 \times \mathbf{H} \to \mathbf{Lag}^{\circ 3} \times \mathbf{H}$, it is clearly smooth and surjective, with connected fibers. It is enough to show that the pull-back $p_{134}^* \left(p_{12}^* \mathcal{K} * p_{23}^* \mathcal{K} \right)$ is irreducible $[n_4]$-perverse. Using Lemma 2 and also invoking some direct diagram chasing one obtains

$$p_{123}^* \left(p_{12}^* \mathcal{K} * p_{23}^* \mathcal{K} \right) \simeq p_{12}^* \mathcal{K} * p_{23}^* \mathcal{K} * p_{34}^* \mathcal{K}. \qquad (11)$$

The right-hand side of (11) is principally a subsequent application of a properly normalized, Fourier transforms on $p_{34}^* \mathcal{K}$, hence by the Katz–Laumon theorem [15] it is irreducible $[n_4]$-perverse.

Let us summarize. We showed that both sheaves $p_{13}^* \mathcal{K}$ and $p_{12}^* \mathcal{K} * p_{23}^* \mathcal{K}$ are irreducible $[n_3]$-perverse and are isomorphic on an open subvariety. This implies that they must be isomorphic. This concludes the proof of the multiplicativity property.

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