Math 491 - Linear Algebra II, Fall 2016

Midterm Preparation

March 15, 2016

<u>Remarks</u>

- Answer all the questions below.
- A definition is just a definition - there is no need to justify it. Just write it down.
- Unless it's a definition, answers should be written in the following format:
  - Write the main points that will appear in your proof of computation. *Main points:...*
  - Write the actual explanation or proof or computation. *Proof:...* or *Computation:...*

1. Direct Sum

   (a) (6) Let V be a vector space of dimension n over a field $\mathbb{F}$. Define what it means for V to be a <u>direct sum</u> of subspaces $V_1, ..., V_k \subset V$.

   (b) (12) Let $V_1, ..., V_k \subset V$ be subspaces of a vector space V. Show that

   $$V = V_1 \oplus \cdots \oplus V_k$$

   if and only if there exist bases $\mathcal{B}_i$ for V_i, for each $1 \leq i \leq n$, such that $\mathcal{B} = \cup_{i=1}^{k} \mathcal{B}_i$ is a basis for V.

   (c) (7) Let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function defined by

   $$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

   for any $x, y \in \mathbb{R}^n$. Let $L \subset \mathbb{R}^n$ be a subspace. Define

   $$L^\perp = \{x \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \text{ for all } y \in L\}.$$

   Show that

   $$\mathbb{R}^n = L \oplus L^\perp.$$

2. Diagonalization

   (a) (6) Let $T : V \to V$ be a linear transformation. Give the geometric definition of what it means for T to be diagonalizable.

   (b) (12) Let $\lambda \in \text{Spec}(T)$ and define its algebraic multiplicity k_λ and geometric multiplicity n_λ. Prove the following theorem:

   Theorem. The transformation T is diagonalizable if and only if

   $$p_T(x) = (x - \lambda_1)^{k_{\lambda_1}} \cdots (x - \lambda_s)^{k_{\lambda_s}},$$

   with $\lambda_i \in \mathbb{F}$ and $k_{\lambda_i} = n_{\lambda_i}$ for each $1 \leq i \leq s$.

   (c) (7) Let $P : V \to V$ be a linear transformation such that $P^2 = P$. Show that P is diagonalizable.

3. Polynomial Rings

   (a) (6) Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ the ring of polynomials over $\mathbb{F}$. Let $f \in \mathbb{F}[x]$ have $\deg(f) \geq 1$. Define what it means for f to be irreducible. State precisely the theorem concerning the decomposition of elements $f \in \mathbb{F}[x]$ into irreducibles.

   (b) (12)Let R be a commutative ring. Define what it means for a ring to be principal.

      (i) Let $\mathbb{F}$ be a field. Show that $\mathbb{F}[x]$ is a PID, i.e., a principal ring and an integral domain.

      (ii) Let $f, g \in \mathbb{F}[x]$. Suppose that f and g have no common irreducible factors. Show that there exists $h, l \in \mathbb{F}[x]$ such that

      $$hf + lg = 1.$$

   (c) (7) Let $f(x) = x^5 + 2 \in \mathbb{R}[x]$. Decompose f into irreducibles in $\mathbb{R}[x]$. Hint: Decompose f explicitly in $\mathbb{C}[x]$ using DeMoivre's Theorem.

4. Cayley–Hamilton Theorem

   (a) (6) Let $T : V \to V$ be a linear transformation. Let $f(x) \in \mathbb{F}[x]$. Explain the meaning of the expression $f(T)$. Show that there exists a nontrivial polynomial $f(x) \in \mathbb{F}[x]$ such that $f(T) = 0$.

   (b) (12) State precisely the Cayley–Hamilton Theorem. Prove the theorem for the transformation $T_A : \mathbb{C}^n \to \mathbb{C}^n$, defined by $T(v) = Av$, with $A \in M_n(\mathbb{C})$. Hint: Repeat the proof given in HW 5 in this specific case.

   (c) (7) Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V over $\mathbb{F}$. Define the minimal polynomial of T and show that the minimal polynomial divides the characteristic polynomial.