Math 491 - Linear Algebra II, Fall 2015

Midterm Preparation

March 24, 2015

<u>Remarks</u>

- Answer all the questions below.
- A definition is just a definition - there is no need to justify it. Just write it down.
- Unless it's a definition, answers should be written in the following format:
  - Write the main points that will appear in your proof of computation. *Main points:...*
  - Write the actual explanation or proof or computation. *Proof:...* or *Computation:...*

1. Direct Sum

   (a) (6) Let V be a vector space of dimension n over a field $\mathbb{F}$. Define what it means for V to be a <u>direct sum</u> of subspaces $V_1, ..., V_k \subset V$.

   (b) (12) Let $V_1, ..., V_k \subset V$ be subspaces of a vector space V. Show that

   $$V = V_1 \oplus \cdots \oplus V_k$$

   if and only if there exist bases $\mathcal{B}_i$ for V_i, for each $1 \leq i \leq n$, such that $\mathcal{B} = \cup_{i=1}^k \mathcal{B}_i$ is a basis for V.

   (c) (7) Let $P : V \to V$ be a linear transformation such that $P^2 = P$. Show that P is diagonalizable.

2. Diagonalization

   (a) (6) Let $T : V \to V$ be a linear transformation. Give the geometric definition of what it means for T to be <u>diagonalizable</u>.

   (b) (12) Let $\lambda \in \mathrm{Spec}(T)$ and define its algebraic multiplicity k_λ and geometric multiplicity n_λ. Prove the following theorem:

<u>Theorem.</u> The transformation T is diagonalizable if and only if

$$p_T(x) = (x-\lambda_1)^{k_{\lambda_1}} \cdots (x-\lambda_s)^{k_{\lambda_s}},$$

with $\lambda_i \in \mathbb{F}$ and $k_{\lambda_i} = n_{\lambda_i}$ for each $1 \leq i \leq s$.

(c) (7) Let $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $T_A(v) = Av$, where

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

For each eigenvalue λ, compute the algebraic and geometric multiplicity of λ. Conclude that T_A is not diagonalizable.

3. Polynomial Rings

   (a) (6) Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ the ring of polynomials over $\mathbb{F}$. Let $f \in \mathbb{F}[x]$ have $\deg(f) \geq 1$. Define what it means for f to be <u>irreducible</u>. State precisely the theorem concerning the decomposition of elements $f \in \mathbb{F}[x]$ into irreducibles.

   (b) (12) Let $f \in \mathbb{C}[x]$ be monic of degree n. Decompose f into a product of irreducibles. Justify your answer.

   (c) (7) Let $f(x) = x^5 + 2 \in \mathbb{R}[x]$. Decompose f into irreducibles in $\mathbb{R}[x]$. Hint: Consider DeMoivre's Theorem.

4. Cayley–Hamilton Theorem

   (a) (6) Let $T : V \to V$ be a linear transformation. Define the characteristic polynomial $p_T(x) \in \mathbb{F}[x]$ of T. Explain what it means to substitute T into $p_T(x)$.

   (b) (12) State precisely the Cayley–Hamilton Theorem. Prove the theorem for the transformation $T_A : \mathbb{C}^n \to \mathbb{C}^n$, defined by $T(v) = Av$, with $A \in M_n(\mathbb{C})$.

   (c) (7) Let $A \in M_3(\mathbb{R})$ be

$$A = \begin{pmatrix} 0 & -3 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

Then $p_A(x) = x^3 - 3x^2 - 3x + 7$. Let

$$f(x) = x^7 - 3x^6 - 2x^5 + 4x^4 - 3x^3 + 7x^2 + x + 1.$$

Compute $f(A)$.