The Vector Space $\mathbb{C}(\mathbb{R})$

Definition: $\mathbb{C}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}\}$ is a vector space over $\mathbb{C}$.

$$\cdot\ (f+g)(t) = f(t) + g(t) \qquad \forall\, \alpha \in \mathbb{C}$$
$$\cdot\ (\alpha \cdot f)(t) = \alpha \cdot f(t) \qquad f,g \in \mathbb{C}(\mathbb{R})$$

Important Linear Transformations:

Time Shift

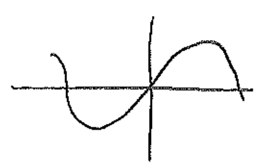
Let $t_0 \in \mathbb{R}$. $L_{t_0} : \mathbb{C}(\mathbb{R}) \longrightarrow \mathbb{C}(\mathbb{R})$

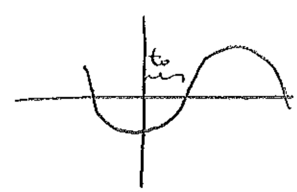
$$f(t) \longmapsto f(t - t_0)$$

L_{t_0} is linear and is in fact an isomorphism with inverse $L_{(-t_0)}$. L_{t_0} is unitary.

Example

For all our examples we will take $f(t) = e^{2\pi i t}$ and we will only graph the imaginary part.





Doppler Shift

Let $w_0 \in \mathbb{R}$. $M_{w_0} : \mathbb{C}(\mathbb{R}) \longrightarrow \mathbb{C}(\mathbb{R})$

$$f(t) \longmapsto e^{2\pi i w_0 t} \cdot f(t)$$

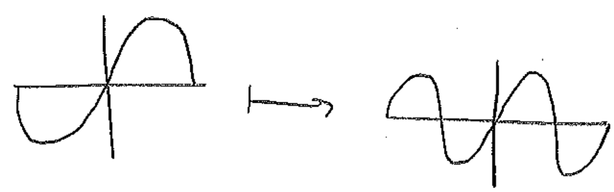
We call this a "shift" because $M_{w_0}(f) = \left[L_{w_0}(\hat{f})\right]^{\vee}$ where $\wedge$ is the fourier transform and $\vee$ its inverse. While M_{w_0} and L_{t_0} do not commute in general we do have the relation,

$$e^{2\pi i w_0 t_0} L_{t_0} M_{w_0} = M_{w_0} L_{t_0}$$

This is known as the Heisenburg commutation relation and will be useful. M_{w_0} is a unitary isomorphism for all w_0.

Example

$w_0 = 1$



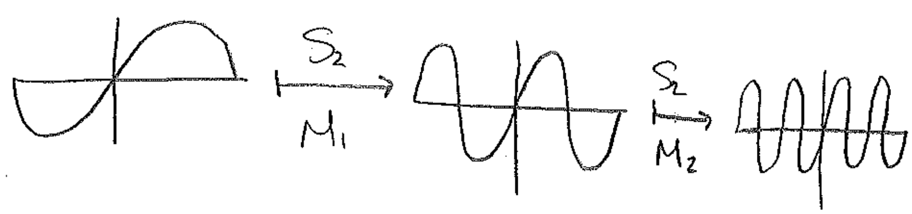
Scaling

Let $a_0 \neq 0 \in \mathbb{R}$. Define $S_{a_0} : C(\mathbb{R}) \longrightarrow C(\mathbb{R})$

$$f(t) \longmapsto f(a_0 t)$$

It is important to note that $\widehat{S_{a_0}(f)} = S_{a_0^{-1}}(\hat{f})$. We will now compare S_{a_0} and M_{w_0}.

Example



So on a plane wave S_{a_0} and M_{w_0} behave similarly but if we are to replace S_{a_0} with M_{w_0} the correct value of w_0 depends on a_0 and the frequency of the wave.

The Channel Model

Let $S(t) \in \mathbb{C}(\mathbb{R})$. If $S(t)$ is transmitted and $R(t)$ is received how can we model the transformation?

$$H: \mathbb{C}(\mathbb{R}) \longrightarrow \mathbb{C}(\mathbb{R})$$

$$S(t) \longmapsto R(t) = \left[\sum_{k=1}^{M} \alpha_k e^{2\pi i w_k t} \cdot S(t-t_k) \right] + W(t)$$

where $\alpha_k, w_k, t_k \in \mathbb{R}$. We will develop this "Delay-Doppler" channel model by considering several physical phenomena.

Assumptions

- Electromagnetic waves propogate at the speed of light, $c \approx 3 \times 10^8$ m/s.
- The power of a signal remains relatively constant.

Time Delay

Because light travels at c m/s it will take any signal t_k seconds to travel a distance of d_k where,

$$t_k = \frac{d_k}{c}.$$

This results in a time shift by t_j seconds.

Attenuation

Transmitting any signal which travels at constant speed results in an inverse square law. The power of a signal (given by its ℓ^2 norm) is constant along concentric spherical shells, and if we integrate the power over any two spherical shells we should obtain the same value. This yields:

$$\|R_j(t)\|_2 \propto \frac{1}{d_j^2}$$

The ℓ^2 norm of the received signal is proportional to $\frac{1}{d_j^2}$ where d_j is again the distance travelled. This means that we will have a power decay which can be modeled by multiplication by $\alpha_j \in \mathbb{R}$ where α_j is proportional to $\frac{1}{d_j}$.

Doppler Effect

Consider moving in the opposite direction of a plane wave $S(t) = e^{2\pi i f t}$ with frequency f.

In one second how many periods will you encounter. You will encounter the f periods you would if stationary and an additional v/λ, where λ is the wavelength of the signal, because you've travelled v meters. Since $\lambda = c/f$ we have

$$f_{eff} = f + \frac{vf}{c} = f(1 + v/c)$$

where f_{eff} is the perceived velocity. So ignoring other effects for the moment we receive;

$$R(t) = M_w[S(t)] = S_a[S(t)]$$

where $w = \frac{vf}{c}$ and $a = (1 + v/c)$. Note that w depends linearly on f. So if we instead transmit $S(t) = \sum_k e^{2\pi i f_k t}$ then we can not model this effect as a Doppler shift.

$$R(t) = S_a[S(t)] \neq M_w[S(t)]$$

Because of certain advantages to modeling Doppler effect as a Doppler shift in certain (most) situations we do. The generally accepted assumption for using a model with Doppler Shift instead of scaling is that,

$$\frac{f_{max} - f_{min}}{f_{min}} \leq 0.05$$

Multipath

Every phenomenon thus far has depended on the path taken by the signal, but the signal in general reaches the receiver along many paths each with its own parameters. Adding in a path-independent white noise arising from other users, physical interference, and other such factors:

$$R(t) = \left[\sum_{k=1}^{M} \alpha_k \cdot e^{2\pi i \omega_k t} \cdot S(t - t_k)\right] + W(t)$$

$$= \left[\sum_{k=1}^{M} \alpha_k \cdot M_{\omega_k} \circ L_{t_k}(S(t))\right] + W(t)$$

Complex Attenuation

For several reasons we want the transmitted signal to occupy a certain "frequency band," $\text{supp}(\hat{s}) \subseteq I$, where I is an interval:

$$\begin{tikzcd} \mathbb{C}(\mathbb{R}) \arrow[r, "H"] & \mathbb{C}(\mathbb{R}) \arrow[d, "\text{Down}"] \\ \mathbb{C}(\mathbb{R}) \arrow[u, "\text{Up}"] \arrow[r, dashed, "H_B"] & \mathbb{C}(\mathbb{R}) \end{tikzcd}$$

- This is how we seperate signals for different uses
- To achieve Delay-Doppler Model assumption
- Physical Properties of certain frequencies
- Sampling Assumptions (Coming Soon)

For this reason we shift our original signal (called the baseband signal) up in frequency, transmit, receive, then shift it back down. We will call the composition of this, H_B, the baseband signal.

The number by which we Doppler Shift our function is called the <u>carrier frequency</u> and is denoted f_c. So if

$$H(S(t)) = \left[\sum_{k=1}^{M} \alpha_k \cdot e^{2\pi i w_k t} \cdot S(t-t_k)\right] + W(t)$$

then for $H_B(S(t)) = e^{2\pi i(-f_c)t} \cdot H(e^{2\pi i f_c t} \cdot S(t))$ we have

$$H_B(S(t)) = e^{-2\pi i f_c t} \cdot \left[\sum_{k=1}^{M} \alpha_k e^{2\pi i f_c (t-t_k)} e^{2\pi i w_c t} S(t-t_k)\right] + W(t)$$

$$= \left[\sum_{k=1}^{M} \alpha_k \cdot e^{-2\pi i f_c t_k} e^{2\pi i w_c t} S(t-t_k)\right] + W(t).$$

This factor $e^{-2\pi i f_c t_k}$ in each term is a unimodular scalar. So we will absorb this scalar into α_k so we now have a complex attenuation. So for H_B we have,

$$H_B(S(t)) = \left[\sum_{k=1}^{M} \alpha_k e^{2\pi i w_k t} S(t-t_k)\right] + W(t)$$

$\alpha_k \in \mathbb{C}$, $w_k \in \mathbb{R}$, $t_k \in \mathbb{R}$.

Goal:

$$\begin{array}{ccc} \mathbb{C}(\mathbb{R}) & \xrightarrow{H} & \mathbb{C}(\mathbb{R}) \\ {\scriptstyle \text{Up}}\uparrow & & \downarrow{\scriptstyle \text{Down}} \\ \mathbb{C}(\mathbb{R}) & \xrightarrow{H_B} & \mathbb{C}(\mathbb{R}) \\ {\scriptstyle \text{D-to-A}}\uparrow & & \downarrow{\scriptstyle \text{A-to-D}} \\ \mathbb{C}(\mathbb{Z}/_N) & \xrightarrow{H_D} & \mathbb{C}(\mathbb{Z}/_N) \end{array}$$

$N \in \mathbb{N}$

$S_D \in \mathbb{C}(\mathbb{Z}/_N)$

Understand H_D in the way we understand H and H_B.

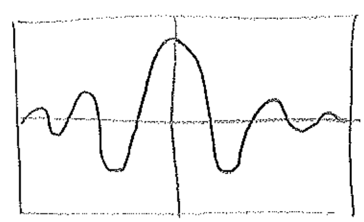
Theorem: Let $W > 0 \in \mathbb{R}$. Suppose $S \in \mathbb{C}(\mathbb{R})$ s.t. $\text{supp}(\hat{S}) \subseteq [\frac{-W}{2}, \frac{W}{2}]$.
Let $S_D \in \mathbb{C}(\mathbb{Z})$ so that $S_D[n] = S(\frac{n}{W})$.

$$S \xmapsto{\circ_D} S_D \quad \text{is one-to-one}$$

Note

For any $f_c \in \mathbb{R}$ the same holds for the interval $[f_c - \frac{W}{2}, f_c + \frac{W}{2}]$.

Definition: $\text{Sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$



- $\text{sinc}(0) = 1$
- $\text{sinc}(n) = 0 \quad \forall\, 0 \neq n \in \mathbb{Z}$.
- $|\text{sinc}(x)| < \frac{1}{x}$
- If $S = \text{sinc}(x)$, then $\hat{S} = \chi_{[-1/2, 1/2]}$
- More generally if $S = \text{sinc}(Wx)$, then $\hat{S} = \chi_{[-W/2, W/2]}$

Conclusion

sinc is the unique function corresponding to $(\ldots 0, 0, 1, 0, 0, \ldots)$

Theorem: There is no function $S \in \mathbb{C}(\mathbb{R})$, $\text{supp}(S) \subseteq [-B, B]$ and $\text{supp}(\hat{S}) \subseteq [-B, B]$ for any $B \in \mathbb{R}$, except 0.

Remark

This provides a theoretical roadblock to what we will be doing. The functions we transmit must leak over in time or frequency.

Definition: Let $S_D \in \mathbb{C}(\mathbb{Z}/N)$ and $W>0$. Define,

$$S_A(t) = \sum_{n=0}^{N-1} S_D[n] \cdot \operatorname{sinc}(Wt-n)$$

We will call the function D-to-A: $\mathbb{C}(\mathbb{Z}/N) \to \mathbb{C}(\mathbb{R})$ that takes $S_D \mapsto S_A$.

Definition: Let $S_A \in \mathbb{C}(\mathbb{R})$, s.t. $\operatorname{supp}(\hat{S_A}) \subseteq [-W/2, W/2]$ and let $N \in \mathbb{N}$. Define,

$$S_D[n] = S_A\left(\frac{n}{W}\right)$$

We will call this function A-to-D: $\mathbb{C}(\mathbb{R}) \to \mathbb{C}(\mathbb{Z}/N)$ that takes $S_A \mapsto S_D$.

Remark

- $(\text{A-to-D}) \circ (\text{D-to-A}) = Id_{\mathbb{C}(\mathbb{Z}/N)}$
- $(\text{D-to-A}) \circ (\text{A-to-D}) \neq Id_{\mathbb{C}(\mathbb{R})}$. Why not?

  In a sense these functions are associating $\mathbb{C}(\mathbb{Z}/N)$ with signals in $\mathbb{C}(\mathbb{R})$ which are band-limited $[-W/2, W/2]$ and "almost" time-limited $[0,T]$. What is T? $T = N/W$.
- Currently this association does not respect time shift.
- Doppler shift is even worse!

$\boxed{\text{Question}}$ How can we solve these problems?

Cyclic Prefixing

Let $t_0 = \frac{\tau_0}{W}$ for some $\tau_0 \in \{0, \ldots, N-1\}$. Suppose H_B is simply

$$S_A \xmapsto{H_B} R_A = S_A(t-t_0)$$

Let's modify the definition of D-to-A. Given $S_D \in \mathbb{C}(\mathbb{Z}/N)$

$$S_A(t) = \sum_{n=-N}^{-1} S_D[n+N] \cdot \mathrm{sinc}(Wt-n)$$
$$+ \sum_{n=0}^{N-1} S_D[n] \cdot \mathrm{sinc}(Wt-n)$$

Now let $R_D = (A\text{-to-}D)(R_A)$.

$$R_D[k] = (A\text{-to-}D)(S_A(t-t_0))$$
$$= S_A\left(\frac{k}{W} - t_0\right) = S_A\left(\frac{k-\tau_0}{W}\right)$$
$$= \sum_{n=-N}^{-1} S_D[n+N] \cdot \mathrm{sinc}(k-\tau_0-n)$$
$$+ \sum_{n=0}^{N-1} S_D[n] \cdot \mathrm{sinc}(k-\tau_0-n)$$
$$= \begin{cases} S_D[k-\tau_0] & k-\tau_0 \geq 0 \\ S_D[k-\tau_0+N] & -N \leq k-\tau_0 < 0 \end{cases}$$

We have achieved the desired effect.

Remark

The same can be done for Doppler shift but often is not, because the Doppler shifts are often quite small.

Digital Doppler Shift

Let $W_o = \frac{\omega_o}{T}$ where $T = N/W$. Consider the simple channel

$$S_A \overset{H_B}{\longmapsto} R_A = e^{2\pi i W_o t} \cdot S_A(t)$$

Beginning with $S_D \in \mathbb{C}(\mathbb{Z}/_N)$ we have:

$$S_A(t) = \sum_{n=0}^{N-1} S_D[n] \operatorname{Sinc}(Wt-n)$$

$$R_A(t) = \sum_{n=0}^{N-1} S_D[n] \cdot e^{2\pi i W_o t} \cdot \operatorname{Sinc}(Wt-n)$$

$$R_D[k] = \sum_{n=0}^{N-1} S_D[n] \cdot e^{2\pi i W_o k/W} \cdot \operatorname{Sinc}(k-n)$$

$$= S_D[k] \cdot e^{\frac{2\pi i \omega_o}{N}}.$$

Multiplication by $e^{\frac{2\pi i \omega_o k}{N}}$ is a Doppler Shift of ω_o on the vector space $\mathbb{C}(\mathbb{Z}/_N)$. The N in the denominator makes functions such as $e^{\frac{2\pi i \omega_o k}{N}}$ periodic for $k \in \mathbb{Z}$.

Conclusion

By these methods we have the following

$$\begin{tikzcd}
\mathbb{C}(\mathbb{R}) \arrow[r, "H_B"] & \mathbb{C}(\mathbb{R}) \arrow[d, "A\text{-to-}D"] \\
\mathbb{C}(\mathbb{Z}/N) \arrow[u, "D\text{-to-}A"] \arrow[r, dashed, "H_D"] & \mathbb{C}(\mathbb{Z}/N)
\end{tikzcd}$$

Suppose $\left[H_B(S_A)\right](t) = \sum_{k=1}^{M} \alpha_k \cdot e^{2\pi i w_k t} \cdot S_A(t - t_k) + W(t)$,

where $\forall k$, $t_k = \frac{\tau_k}{W}$, $w_k = \frac{\omega_k}{T}$ then what we have is,

$$\begin{aligned}
\left[H_D(S_D)\right][n] &= \left[(A\text{-to-}D) \circ H_B \circ (D\text{-to-}A)\right](S_D) \\
&= \left[\sum_{k=1}^{M} \alpha_k \cdot e^{2\pi i \omega_k n} \cdot S_D[n - \tau_k]\right] + W[n]
\end{aligned}$$