

# On Some Deterministic Dictionaries Supporting Sparsity

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Received: 28 September 2007 / Revised: 26 April 2008  
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**Abstract** We describe a new construction of an incoherent dictionary, referred to as the *oscillator dictionary*, which is based on considerations in the representation theory of finite groups. The oscillator dictionary consists of approximately  $p^5$  unit vectors in a Hilbert space of dimension  $p$ , whose pairwise inner products have magnitude of at most  $4/\sqrt{p}$ . An explicit algorithm to construct a large portion of the oscillator dictionary is presented.

**Keywords** Sparsity · Deterministic dictionaries · Low coherence · Weil representation · Commutative subgroups · Eigenfunctions · Explicit algorithm

**Mathematics Subject Classification (2000)** 94A12 · 11F27

## 1 Introduction

Digital signals, or simply signals, can be thought of as functions on the finite line  $\mathbb{F}_p$ , namely the finite field with  $p$  elements, where  $p$  is a prime number. The space of

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Communicated by Ronald A. DeVore.

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signals  $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$  is a Hilbert space, with the inner product given by the standard formula

$$\langle f, g \rangle = \sum_{t \in \mathbb{F}_p} f(t) \overline{g(t)}.$$

### 1.1 Incoherent Dictionaries

A central problem is to construct useful classes of signals that demonstrate strong descriptive power and at the same time are characterized by formal mathematical conditions. Meeting these two requirements is a non-trivial task and is a source for many novel developments in the field of signal processing. The problem was tackled, over the years, by various approaches.

Two decades ago [4], a novel approach was introduced, hinting towards a fundamental change of perspective about the nature of signals. In this approach, a signal is characterized in terms of its sparsest presentation as a linear combination of vectors (also called *atoms*) in a *dictionary*. The characterization is intrinsically non-linear, hence, as a consequence, one comes to deal with classes of signals which are not closed with respect to addition. More formally:

**Definition 1.1** A set of vectors  $\mathcal{D} \subset \mathcal{H}$  is called an *N-independent dictionary* if every subset  $\mathcal{D}' \subset \mathcal{D}$ , with  $|\mathcal{D}'| = N$ , is linearly independent.

This notion is very close to the notion of *spark* of a dictionary introduced in [5].

Given an  $2N$ -independent dictionary  $\mathcal{D}$ , every signal  $f \in \mathcal{H}$ , has at most one presentation of the form

$$f = \sum_{\varphi \in \mathcal{D}'} a_{\varphi} \varphi,$$

for  $\mathcal{D}' \subset \mathcal{D}$  with  $|\mathcal{D}'| \leq N$ . Such a presentation, if exists, is unique and is called the *sparse* presentation. Consequently, we will also call such a dictionary  $\mathcal{D}$  an *N-sparse* dictionary. Given that a signal  $f$  admits a sparse presentation, a basic difficulty is to effectively reconstruct the sparse coefficients  $a_{\varphi}$ . A way to overcome this difficulty is to introduce [3, 5–9, 18] the stronger notion of *incoherent dictionary*.

**Definition 1.2** A set of vectors  $\mathcal{D} \subset \mathcal{H}$  is called  *$\mu$ -coherent* dictionary, for  $0 \leq \mu \ll 1$ , if for every two different vectors  $\varphi, \phi \in \mathcal{D}$  we have  $|\langle \varphi, \phi \rangle| \leq \mu$ .

The two notions of coherence and sparsity are related by the following proposition [3, 5, 6, 9, 18].

**Theorem 1.3** If  $\mathcal{D}$  is  $1/R$ -coherent then  $\mathcal{D}$  is  $\lfloor R/2 \rfloor$ -sparse,<sup>1</sup> moreover there exists an effective algorithm to extract the sparse coefficients.

<sup>1</sup>Here  $\lfloor R/2 \rfloor$  stands for the greatest integer which is less then or equal to  $R/2$ .

A basic problem [3, 4, 14, 16] in the theory is introducing systematic constructions of “good” incoherent dictionaries. Here “good” means that the size of the dictionary and the sparsity factor  $N$  are made as large as possible.

In this paper, we begin to develop a systematic approach to the construction of incoherent dictionaries based on the representation theory of groups over finite fields. In particular, we describe an examples of such dictionary called the oscillator dictionary.

## 1.2 Main Results

The main contribution of this paper is the introduction of a dictionary  $\mathfrak{D}_O$ , that we call the *oscillator* dictionary, which is constructed using the representation theory of the two-dimensional symplectic group  $SL_2(\mathbb{F}_p)$ . The oscillator dictionary is  $4/\sqrt{p}$ -coherent, consisting of approximately  $p^3$  vectors. We also introduce an extended oscillator dictionary  $\mathfrak{D}_E$  which is  $4/\sqrt{p}$ -coherent and consists of approximately  $p^5$  vectors. Our goal is to explain the construction of  $\mathfrak{D}_O$  and state some of its properties which are relevant to sparsity, referring the reader to [11] for a more comprehensive treatment.

As a suggestive model example we explain first the construction of the well known *Heisenberg* dictionary  $\mathfrak{D}_H$  (see [12, 13]), which is constructed using the representation theory of the finite *Heisenberg group* over the finite field  $\mathbb{F}_p$ . The Heisenberg dictionary is  $1/\sqrt{p}$ -coherent, consisting of approximately  $p^2$  vectors.

## 1.3 Structure of the Paper

The paper consists of two sections and two appendices. In Sect. 2, several basic notions from representation theory are introduced. Particularly, we present the Heisenberg and Weil representations over finite fields. In Sect. 3, we introduce the Heisenberg and oscillator dictionaries  $\mathfrak{D}_H$  and  $\mathfrak{D}_O$  respectively, and the extended dictionary  $\mathfrak{D}_E$ . In Appendix A, we explain in more details basic concepts from group representation theory that we use in the body of the paper. Finally, in Appendix B, we describe an explicit algorithm that generates a large portion of the oscillator dictionary.

*Remark 1.4* (Field extension) All the results in this paper were stated for the basic finite field  $\mathbb{F}_p$ , for the reason of making the terminology more accessible. However, they are valid [11] for any field extension of the form  $\mathbb{F}_q$  with  $q = p^n$ . One should only replace  $p$  by  $q$  in all appropriate places.

## 2 The Heisenberg and Weil Representations

### 2.1 The Heisenberg Group

Let  $(V, \omega)$  be a two-dimensional symplectic vector space over the finite field  $\mathbb{F}_p$ . The reader should think of  $V$  as  $\mathbb{F}_p \times \mathbb{F}_p$  with the standard symplectic form

$$\omega((\tau, w), (\tau', w')) = \tau w' - w \tau'.$$

Considering  $V$  as an Abelian group, it admits a non-trivial central extension called the *Heisenberg group*. Concretely, the group  $H$  can be presented as the set  $H = V \times \mathbb{F}_p$  with the multiplication given by

$$(v, z) \cdot (v', z') = \left( v + v', z + z' + \frac{1}{2}\omega(v, v') \right).$$

The center of  $H$  is  $Z = Z(H) = \{(0, z) : z \in \mathbb{F}_p\}$ . The symplectic group  $Sp = Sp(V, \omega)$ , which in this case is just isomorphic to  $SL_2(\mathbb{F}_p)$ , acts by automorphism of  $H$  through its action on the  $V$ -coordinate, that is, a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

sends an element  $(v, z)$ , where  $v = (\tau, w)$  to the element  $(gv, z)$  where  $gv = (a\tau + bw, c\tau + dw)$ .

## 2.2 The Heisenberg Representation

One of the most important attributes of the group  $H$  is that it admits, principally, a unique irreducible representation (see Appendix A.2). The precise statement goes as follows: Let  $\psi : Z \rightarrow S^1$ , where  $S^1$  denotes the unit circle, be a non-trivial unitary character of the center, that is  $\psi \neq 1$  and satisfies  $\psi(z_1 + z_2) = \psi(z_1) \cdot \psi(z_2)$ , for every  $z_1, z_2 \in Z$ ; for example, in this paper we take  $\psi(z) = e^{\frac{2\pi i}{p}z}$ .

We denote by  $U(\mathcal{H})$  the group of unitary operators on  $\mathcal{H}$ . It is not difficult to show [17] that

**Theorem 2.1** (Stone-von Neuman) *There exists a unique (up to isomorphism) irreducible unitary representation  $\pi : H \rightarrow U(\mathcal{H})$  with central character  $\psi$ , that is,  $\pi(z) = \psi(z) \cdot \text{Id}_{\mathcal{H}}$ , for every  $z \in Z$ .*

The representation  $\pi$  which appears in the above theorem will be called the *Heisenberg representation*.

More concretely,  $\pi : H \rightarrow U(\mathcal{H})$  can be realized as follows:  $\mathcal{H}$  is the Hilbert space  $\mathbb{C}(\mathbb{F}_p)$  of complex valued functions on the finite line, with the standard inner product

$$\langle f, g \rangle = \sum_{t \in \mathbb{F}_p} f(t) \overline{g(t)},$$

for every  $f, g \in \mathbb{C}(\mathbb{F}_p)$  and the action  $\pi$  is given by:

- $\pi(\tau, 0)[f](t) = f(t + \tau)$ ;
- $\pi(0, w)[f](t) = \psi(wt)f(t)$ ;
- $\pi(z)[f](t) = \psi(z)f(t)$ ,  $z \in Z$ .

Here we are using  $\tau$  to indicate the first coordinate and  $w$  to indicate the second coordinate of  $V \simeq \mathbb{F}_p \times \mathbb{F}_p$ .

We will call this explicit realization the *standard realization*.

### 2.3 The Weil Representation

A direct consequence of Theorem 2.1 is the existence of a projective unitary representation  $\tilde{\rho} : Sp \rightarrow U(\mathcal{H})$ , that is, a collection of operators  $\{\tilde{\rho}(g) \in U(\mathcal{H}) : g \in Sp\}$  which satisfy multiplicativity up-to a unitary scalar

$$\tilde{\rho}(gh) = C(g, h) \cdot \tilde{\rho}(g) \circ \tilde{\rho}(h),$$

for every  $g, h \in Sp$  and  $C(g, h) \in S^1$ . The construction of  $\tilde{\rho}$  out of the Heisenberg representation  $\pi$  is due to Weil [19] and it goes as follows: Considering the Heisenberg representation  $\pi : H \rightarrow U(\mathcal{H})$  and an element  $g \in Sp$ , one can define a new representation  $\pi^g : H \rightarrow U(\mathcal{H})$  by  $\pi^g(h) = \pi(g(h))$ . Clearly both  $\pi$  and  $\pi^g$  have the same central character  $\psi$  hence, by Theorem 2.1, they are isomorphic. Since the space of intertwining morphisms (see Appendix A.4)  $\text{Hom}_H(\pi, \pi^g)$  is one-dimensional (this follows from Schur's lemma, see Appendix A.2), choosing for every  $g \in Sp$  a non-zero representative  $\tilde{\rho}(g) \in \text{Hom}_H(\pi, \pi^g)$  gives the required projective representation.

In more concrete terms, the projective representation  $\tilde{\rho}$  is characterized by the formula

$$\tilde{\rho}(g) \pi(h) \tilde{\rho}(g^{-1}) = \pi(g(h)), \quad (2.1)$$

for every  $g \in Sp$  and  $h \in H$ .

The important and non-trivial statement is that the projective representation  $\tilde{\rho}$  can be linearized in a unique manner into an honest unitary representation:

**Theorem 2.2** *There exists a unique<sup>2</sup> unitary representation*

$$\rho : Sp \longrightarrow U(\mathcal{H}),$$

such that every operator  $\rho(g)$  satisfies (2.1).

For the sake of concreteness, let us give an explicit description of the operators  $\rho(g)$ , for different elements  $g \in Sp$ , as they appear in the standard realization. The operators will be specified up to a unitary scalar.

- The standard diagonal subgroup  $A \subset Sp$  acts by (normalized) scaling: An element

$$a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

acts by

$$S_a[f](t) = \sigma(a) f(a^{-1}t),$$

where  $\sigma : \mathbb{F}_p^\times \rightarrow \{\pm 1\}$  is the unique non-trivial quadratic character of the multiplicative group  $\mathbb{F}_p^\times$  (also called the Legendre character), given by  $\sigma(a) = a^{\frac{p-1}{2}} \pmod{p}$ .

<sup>2</sup>Unique, except in the case the finite field is  $\mathbb{F}_3$ .

- The subgroup of strictly lower diagonal elements  $U \subset Sp$  acts by quadratic exponents (chirps): An element

$$u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

acts by

$$M_u[f](t) = \psi\left(-\frac{u}{2}t^2\right)f(t).$$

- The Weyl element

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

acts by discrete Fourier transform

$$F[f](w) = \frac{1}{\sqrt{p}} \sum_{t \in \mathbb{F}_p} \psi(wt) f(t).$$

### 3 The Heisenberg and the Oscillator Dictionaries

#### 3.1 Model Example: The Heisenberg Dictionary

The Heisenberg dictionary is a collection of  $p + 1$  orthonormal bases, each characterized, roughly, as eigenvectors of a specific linear operator. An elegant way to define this dictionary is using the Heisenberg representation [12, 13].

##### 3.1.1 Bases Associated with Lines

The Heisenberg group is non-commutative, yet it consists of various commutative subgroups which can be easily described as follows: Let  $L \subset V$  be a line through the origin in  $V$ . One can associate to  $L$  a commutative subgroup  $A_L \subset H$ , given by  $A_L = \{(l, 0) : l \in L\}$ . It will be convenient to identify the group  $A_L$  with the line  $L$ . Restricting the Heisenberg representation  $\pi$  to the commutative subgroup  $L$ , namely, considering the restricted representation  $\pi : L \rightarrow U(\mathcal{H})$ , one obtains a collection of pairwise commuting operators  $\{\pi(l) : l \in L\}$ , which, in turns, yields an orthogonal decomposition into character spaces (see Appendix A.5)

$$\mathcal{H} = \bigoplus_{\chi} \mathcal{H}_{\chi},$$

where  $\chi$  runs in the set  $\widehat{L}$  of unitary characters of  $L$ , that is, each  $\chi \in \widehat{L}$  is a function  $\chi : L \rightarrow S^1$  which satisfies  $\chi(l_1 + l_2) = \chi(l_1) \cdot \chi(l_2)$ , for every  $l_1, l_2 \in L$ .

A more concrete way to specify the above decomposition is by choosing a non-zero vector  $l_0 \in L$ . After such a choice, the character space  $\mathcal{H}_{\chi}$  naturally corresponds to the eigenspace of the linear operator  $\pi(l_0)$  associated with the eigenvalue  $\lambda = \chi(l_0)$ .

It is not difficult to verify in this case that

**Lemma 3.1** For every  $\chi \in \widehat{L}$  we have  $\dim \mathcal{H}_\chi = 1$ .

Choosing a vector  $\varphi_\chi \in \mathcal{H}_\chi$  of norm  $\|\varphi_\chi\| = 1$ , for every  $\chi \in \widehat{L}$  which appears in the decomposition, we obtain an orthonormal basis which we denote by  $B_L$ .

Since there exist  $p + 1$  different lines in  $V$ , we obtain in this manner a collection of  $p + 1$  orthonormal bases, overall constructing a dictionary of vectors  $\mathfrak{D}_H = \{\varphi \in B_L : L \subset V\}$  consisting of  $p(p + 1)$  vectors. We will call this dictionary, for obvious reasons, the *Heisenberg dictionary*.

The main property of the Heisenberg dictionary is summarized in the following theorem [12, 13].

**Theorem 3.2** For every pair of different lines  $L, M \subset V$  and for every  $\varphi \in B_L$ ,  $\phi \in B_M$

$$|\langle \varphi, \phi \rangle| = \frac{1}{\sqrt{p}}.$$

### 3.1.2 The Standard Bases

There are two standard examples of bases of the form  $B_L$  associated with the standard lines  $T = \{(\tau, 0) : \tau \in \mathbb{F}_p\}$  and  $W = \{(0, w) : w \in \mathbb{F}_p\}$ . The basis  $B_W$  consists of delta functions  $\delta_a$ ,  $a \in \mathbb{F}_p$ , i.e.,  $\delta_a(t) = 1$  if  $a = t$  and  $\delta_a(t) = 0$  otherwise, and the basis  $B_T$  consists of normalized characters  $\psi_a$ ,  $a \in \mathbb{F}_p$ , where  $\psi_a(t) = 1/\sqrt{p}\psi(at)$ .

Indeed, the delta functions are common eigenfunctions of the operators  $\pi(0, w)$ ,  $w \in \mathbb{F}_p$  and the characters are common eigenfunctions of the operators  $\pi(\tau, 0)$ ,  $\tau \in \mathbb{F}_p$ .

Finally, for this specific example, the assertion of Theorem 3.2 amounts to

$$|\langle \delta_a, \psi_b \rangle| = \frac{1}{\sqrt{p}}, \quad (3.1)$$

for every  $\delta_a \in B_W$  and  $\psi_b \in B_T$ .

Theorem 3.2 asserts that (3.1) holds for the larger collection  $\mathfrak{D}_H$  of  $p + 1$  orthonormal bases.

## 3.2 The Oscillator Dictionary

Reflecting back on the Heisenberg dictionary we see that it consists of a collection of orthonormal bases characterized in terms of commutative families of unitary operators where each such family is associated with a commutative subgroup in the Heisenberg group  $H$ , via the Heisenberg representation  $\pi : H \rightarrow U(\mathcal{H})$ . In comparison, the oscillator dictionary [11] will be characterized in terms of commutative families of unitary operators which are associated with commutative subgroups in the symplectic group  $Sp$  via the Weil representation  $\rho : Sp \rightarrow U(\mathcal{H})$ .

### 3.2.1 Maximal Tori

The commutative subgroups in  $Sp$  that we consider are called maximal algebraic tori [2] (not to be confused with the notion of a topological torus). A maximal (algebraic) torus in  $Sp$  is a maximal commutative subgroup which becomes diagonalizable over some field extension. The most standard example of a maximal algebraic torus is the standard diagonal torus

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^\times \right\}.$$

Standard linear algebra shows that up to conjugation<sup>3</sup> there exist two classes of maximal (algebraic) tori in  $Sp$ . The first class consists of those tori which are diagonalizable already over  $\mathbb{F}_p$ , namely, those are tori  $T$  which are conjugated to the standard diagonal torus  $A$  or more precisely such that there exists an element  $g \in Sp$  so that  $g \cdot T \cdot g^{-1} = A$ . A torus in this class is called a *split* torus.

The second class consists of those tori which become diagonalizable over the quadratic extension  $\mathbb{F}_{p^2}$ , namely, those are tori which are not conjugated to the standard diagonal torus  $A$ . A torus in this class is called a *non-split* torus (sometimes it is called inert torus).

All split (non-split) tori are conjugated to one another, therefore the number of split tori is the number of elements in the coset space  $Sp/N$  (see [1] for basics of group theory), where  $N$  is the normalizer group of  $A$ ; we have

$$\#(Sp/N) = \frac{p(p+1)}{2},$$

and the number of non-split tori is the number of elements in the coset space  $Sp/M$ , where  $M$  is the normalizer group of some non-split torus; we have

$$\#(Sp/M) = p(p-1).$$

*Example of a Non-split Maximal Torus* It might be suggestive to explain further the notion of non-split torus by exploring, first, the analogue notion in the more familiar setting of the field  $\mathbb{R}$ . Here, the standard example of a maximal non-split torus is the circle group  $SO(2) \subset SL_2(\mathbb{R})$ . Indeed, it is a maximal commutative subgroup which becomes diagonalizable when considered over the extension field  $\mathbb{C}$  of complex numbers. The above analogy suggests a way to construct examples of maximal non-split tori in the finite field setting as well.

Let us assume for simplicity that  $-1$  does not admit a square root in  $\mathbb{F}_p$  or equivalently that  $p \equiv 1 \pmod{4}$ . The group  $Sp$  acts naturally on the plane  $V = \mathbb{F}_p \times \mathbb{F}_p$ . Consider the standard symmetric form  $B$  on  $V$  given by

$$B((\tau, w), (\tau', w')) = \tau\tau' + ww'.$$

<sup>3</sup>Two elements  $h_1, h_2$  in a group  $G$  are called conjugated elements if there exists an element  $g \in G$  such that  $g \cdot h_1 \cdot g^{-1} = h_2$ . More generally, two subgroups  $H_1, H_2 \subset G$  are called conjugated subgroups if there exists an element  $g \in G$  such that  $g \cdot H_1 \cdot g^{-1} = H_2$ .



An example of maximal non-split torus is the subgroup  $SO = SO(V, B) \subset Sp$  consisting of all elements  $g \in Sp$  preserving the form  $B$ , namely  $g \in SO$  if and only if  $B(gu, gv) = B(u, v)$  for every  $u, v \in V$ . In coordinates,  $SO$  consists of all matrices  $A \in SL_2(\mathbb{F}_p)$  which satisfy  $AA^t = I$ . The reader might think of  $SO$  as the “finite circle”.

### 3.2.2 Bases Associated with Maximal Tori

Restricting the Weil representation to a maximal torus  $T \subset Sp$ , one obtains a representation of a commutative group  $\rho : T \rightarrow U(\mathcal{H})$ , which, in turns, yields an orthogonal decomposition into character spaces (see Appendix A.5)

$$\mathcal{H} = \bigoplus_{\chi} \mathcal{H}_{\chi}, \quad (3.2)$$

where  $\chi$  runs in the set  $\widehat{T}$  of unitary characters of the torus  $T$ , that is, each  $\chi$  is a function  $\chi : T \rightarrow S^1$ , satisfying  $\chi(t_1 \cdot t_2) = \chi(t_1) \cdot \chi(t_2)$ , for every  $t_1, t_2 \in T$ .

A more concrete way to specify the above decomposition is by choosing a generator<sup>4</sup>  $t_0 \in T$ , that is, an element such that every  $t \in T$  can be written in the form  $t = t_0^n$ , for some  $n \in \mathbb{N}$ . After such a choice, the character space  $\mathcal{H}_{\chi}$  which appears in (3.2) naturally corresponds to the eigenspace of the linear operator  $\rho(t_0)$  associated to the eigenvalue  $\lambda = \chi(t_0)$ .

The decomposition (3.2) depends on the type of  $T$ :

- In the case where  $T$  is a split torus we have  $\dim \mathcal{H}_{\chi} = 1$  unless  $\chi = \sigma$ , where  $\sigma : T \rightarrow \{\pm 1\}$  is the unique non-trivial quadratic character of  $T$  (also called the *Legendre* character of  $T$ ), in the latter case  $\dim \mathcal{H}_{\sigma} = 2$ .
- In the case where  $T$  is a non-split torus then  $\dim \mathcal{H}_{\chi} = 1$  for every character  $\chi$  which appears in the decomposition, in this case the quadratic character  $\sigma$  does not appear in the decomposition (for details see [10]).

Choosing for every character  $\chi \in \widehat{T}$ ,  $\chi \neq \sigma$ , a vector  $\varphi_{\chi} \in \mathcal{H}_{\chi}$  of unit norm, we obtain an orthonormal system of vectors  $B_T = \{\varphi_{\chi} : \chi \neq \sigma\}$ , noting that in the case when  $T$  is a non-split torus, the set  $B_T$  is, in fact, an orthonormal basis. Considering the union of all these systems, we obtain the oscillator dictionary

$$\mathcal{D}_O = \{\varphi \in B_T : T \subset Sp\}.$$

It is convenient to separate the dictionary  $\mathcal{D}_O$  into two sub-dictionaries  $\mathcal{D}_O^s$  and  $\mathcal{D}_O^{ns}$  which correspond to the split tori and the non-split tori respectively. The split sub-dictionary  $\mathcal{D}_O^s$  consists of the union of all orthonormal systems  $B_T$ , where  $T$  runs through all the split tori in  $Sp$ , altogether  $\frac{p(p+1)}{2}$  such systems, each consisting of  $p - 2$  orthonormal vectors, hence

$$\#\mathcal{D}_O^s = \frac{p(p+1)(p-2)}{2}.$$

<sup>4</sup>A maximal torus  $T$  in  $SL_2(\mathbb{F}_p)$  is a cyclic group, thus there exists a generator.

The non-split sub-dictionary  $\mathfrak{D}_O^{ns}$  consists of the union of all orthonormal bases  $B_T$ , where  $T$  runs through all the non-split tori in  $Sp$ , altogether  $p(p-1)$  such bases, each consisting of  $p$  orthonormal vectors, hence

$$\#\mathfrak{D}_O^{ns} = p^2(p-1).$$

Vectors in the oscillator dictionary satisfy many desired properties [11]. In this paper we are only interested in the following property:

**Theorem 3.3** [11] *Let  $\phi \in B_{T_1}$  and  $\varphi \in B_{T_2}$*

$$|\langle \phi, \varphi \rangle| \leq \frac{4}{\sqrt{p}}.$$

*The System Associated with the Standard Torus* It would be beneficial to give an explicit description of the system  $B_A$  where  $A \subset Sp$  is the standard diagonal torus, which is isomorphic to the multiplicative group  $G_m = \mathbb{F}_p^\times$ . The torus  $A$  acts on the Hilbert space  $\mathcal{H}$  via the Weil representation yielding a decomposition into character spaces

$$\mathcal{H} = \bigoplus_{\chi \in \widehat{A}} \mathcal{H}_\chi.$$

For every  $\chi \neq \sigma$  the character space  $\mathcal{H}_\chi$  is one-dimensional. Our goal is to describe an explicit vector  $\varphi_\chi \in \mathcal{H}_\chi$  of unit norm: Let  $\chi : G_m \rightarrow S^1$  be a non-trivial ( $\chi \neq 1$ ) unitary character of the multiplicative group. Thinking of the multiplicative group  $G_m = \mathbb{F}_p^\times$  as sitting inside the line  $\mathbb{F}_p$  we define the function  $\varphi_\chi \in \mathbb{C}(\mathbb{F}_p)$  as follows:

$$\varphi_\chi(t) = \begin{cases} \frac{1}{\sqrt{p-1}} \chi(t), & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Since, for every  $a \in A$ ,  $\rho(a)$  acts by normalized scaling (see Sect. 2.3), it is easy to verify that  $\varphi_\chi$  is a character vector with respect to the action  $\rho : A \rightarrow U(\mathcal{H})$  associated to the character  $\chi \cdot \sigma$ .

Concluding, the orthonormal system  $B_A$  is the set  $\{\varphi_\chi : \chi \in \widehat{G}_m, \chi \neq 1\}$ .

### 3.2.3 Extended Oscillator Dictionary

The oscillator dictionary can be extended to a much larger dictionary  $\mathfrak{D}_E$  using the action of the Heisenberg group. Given a vector  $\varphi \in \mathfrak{D}_O$  one can consider its orbit under the action of the set  $V \subset H$

$$\mathcal{O}_\varphi = V \cdot \varphi \triangleq \{\pi(v)\varphi : v \in V\}.$$

It is not hard to show that orbits associated to different vectors are disjoint, therefore, we obtain a dictionary

$$\mathfrak{D}_E = \bigcup_{\varphi \in \mathfrak{D}} \mathcal{O}_\varphi,$$

consisting of  $\#(V) \cdot \#(\mathfrak{D}_O) \sim p^5$  vectors. Interestingly, the extended dictionary  $\mathfrak{D}_E$  continues to be  $4/\sqrt{p}$ -coherent, this is a consequence of the following generalization of Theorem 3.3.

**Theorem 3.4** [10] *Given two vectors  $\varphi, \phi \in \mathfrak{D}_O$  and an element  $0 \neq v \in V$  we have*

$$|\langle \varphi, \pi(v)\phi \rangle| \leq \frac{4}{\sqrt{p}}.$$

For a proof, see [11].

**Remark 3.5** A way to interpret Theorem 3.4 is to say that any two different vectors  $\varphi \neq \phi \in \mathfrak{D}_O$  are incoherent in a stable sense, that is, their coherency is  $4/\sqrt{p}$  no matter if any one of them undergoes an arbitrary time/phase shift. This property seems to be important in communication where a transmitted signal may acquire time shift due to asynchronous communication and phase shift due to Doppler effect.

**Acknowledgement** It is a pleasure to thank J. Bernstein for his interest and guidance in the mathematical aspects of this work. We are grateful to S. Golomb and G. Gong for their interest in this project. We would like to thank M. Elad, O. Holtz, R. Kimmel, L.H. Lim, and A. Sahai for interesting discussions. Finally, we thank B. Sturmfels for encouraging us to proceed in this line of research.

## Appendix A: Terminology from Representation Theory

### A.1 Finite Fields

Given a prime number  $p \geq 2$ , there exists a unique finite field consisting of  $p$  elements, denoted by  $\mathbb{F}_p$ . A way to visualize this field is as a discrete set of  $p$  points, cyclically ordered and indexed by the numbers  $0, 1, \dots, p-1$ .

Most of the constructions of linear algebra carry over to the finite field setting, in particular one can consider matrix groups with matrix entries from  $\mathbb{F}_p$ . Particular examples of such groups which play a role in this paper are the special linear group  $SL_2(\mathbb{F}_p)$  and the special orthogonal group  $SO_2(\mathbb{F}_p)$ . The first, consists of  $2 \times 2$  matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{F}_p,$$

such that  $ad - bc = 1$ . The second is a subgroup of  $SL_2(\mathbb{F}_p)$  consisting of matrices  $A \in SL_2(\mathbb{F}_p)$  such that  $AA^t = \text{Id}$ .

### A.2 Unitary Representations

Let  $\mathcal{H}$  be a finite dimensional complex Hilbert space, equipped with an inner product  $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ . A unitary operator on  $\mathcal{H}$  is an operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  which preserves the inner product, that is,  $\langle Af, Ag \rangle = \langle f, g \rangle$ , for every  $f, g \in \mathcal{H}$ . The set of unitary operators forms a group under composition of operators, which is denoted by  $U(\mathcal{H})$ .

We proceed to introduce the notion of a unitary representation (see [1, 15] for a more comprehensive treatment). Let  $G$  be a finite group.

**Definition A.1** A unitary representation of  $G$  on the Hilbert space  $\mathcal{H}$  is a homomorphism  $\pi : G \rightarrow U(\mathcal{H})$ , that is,  $\pi$  is a map which satisfies the condition

$$\pi(g \cdot h) = \pi(g) \circ \pi(h),$$

for every  $g, h \in G$ .

Specifying a unitary representation  $\pi : G \rightarrow U(\mathcal{H})$  gives a convenient way to think of the collection of unitary operators  $\{\pi(g) : g \in G\}$  and the relations that they satisfy between one another—these relations are encoded in the structure of the group  $G$ .

The smallest unitary representations are the *irreducible* unitary representations.

**Definition A.2** A unitary representation  $\pi : G \rightarrow U(\mathcal{H})$  is called irreducible if there is no proper vector space  $0 \neq \mathcal{H}' \subsetneq \mathcal{H}$  invariant under  $G$ , i.e., such that

$$\pi(g)[f] \in \mathcal{H}',$$

for every  $f \in \mathcal{H}'$ .

Irreducible unitary representations form the building blocks of all unitary representations in the sense that every unitary representation  $\pi : G \rightarrow U(\mathcal{H})$  can be decomposed into a direct sum of irreducible unitary representations. The precise statement is that always there exists a decomposition of the Hilbert space  $\mathcal{H}$  into a direct sum

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i,$$

such that each subspace  $\mathcal{H}_i$  is closed under the action of  $G$ , that is  $\pi(g)[f] \in \mathcal{H}_i$ , for every  $f \in \mathcal{H}_i$  and such that the restricted unitary representations  $\pi_i : G \rightarrow U(\mathcal{H}_i)$  are irreducible.

### A.2.1 Unitary Representations of Commutative Groups

A particular situation occurs when  $G$  is a commutative group, that is, a group for which  $g \cdot h = h \cdot g$ , for every  $g, h \in G$ . In this situation, specifying a unitary representation  $\pi : G \rightarrow U(\mathcal{H})$  is equivalent to specifying a collection of unitary operators  $\{\pi(g) : g \in G\}$  which commute pairwise; this follows from the following relation:

$$\pi(g) \circ \pi(h) = \pi(g \cdot h) = \pi(h \cdot g) = \pi(h) \circ \pi(g),$$

for every  $g, h \in G$ .

### A.3 Basic Examples

We proceed to describe three basic examples of unitary representations of commutative groups, which are of particular relevance to this paper. In all these examples the Hilbert space is taken to be  $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$ .

#### A.3.1 Time Shifts

Let  $(\mathbb{F}_p, +)$  be the additive group, let us denote the parameter of  $\mathbb{F}_p$  by  $\tau$ . Define the unitary representation  $L : \mathbb{F}_p \rightarrow U(\mathcal{H})$  given by  $\tau \mapsto L_\tau$ , where  $L_\tau$  is the unitary operator of cyclic time translation by  $\tau$ :

$$L_\tau[f](t) = f(t + \tau),$$

for every  $f \in \mathcal{H}$ .

#### A.3.2 Phase Shifts

Let  $(\mathbb{F}_p, +)$  be the same as in the previous example, let us denote the parameter of  $\mathbb{F}_p$  by  $w$ . Define the unitary representation  $M : \mathbb{F}_p \rightarrow U(\mathcal{H})$  given by  $w \mapsto M_w$ , where  $M_w$  is the unitary operator of cyclic phase translation by  $w$ :

$$M_w[f](t) = e^{\frac{2\pi i}{p} wt} f(t),$$

for every  $f \in \mathcal{H}$ .

#### A.3.3 Scaling

Let  $(\mathbb{F}_p^\times, \cdot)$  be the multiplicative group, let us denote the parameter of  $\mathbb{F}_p^\times$  by  $a$ . Define the unitary representation  $S : \mathbb{F}_p^\times \rightarrow U(\mathcal{H})$  given by  $a \mapsto S_a$ , where  $S_a$  is the unitary operator of scaling by  $a$ :

$$S_a[f](t) = f(a \cdot t),$$

for every  $f \in \mathcal{H}$ .

### A.4 Intertwining Morphisms

Let  $\pi_i : G \rightarrow U(\mathcal{H}_i)$ ,  $i = 1, 2$ , be a pair of unitary representations.

**Definition A.3** An intertwining morphism from  $\pi_1$  to  $\pi_2$  is a unitary operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  which satisfies

$$A \circ \pi_1(g)[f] = \pi_2(g) \circ A[f],$$

for every  $g \in G$  and for every  $f \in \mathcal{H}_1$ .

The space of intertwining morphisms from  $\pi_1$  to  $\pi_2$  is denoted by  $\text{Hom}_G(\pi_1, \pi_2)$ . In addition, if there exist  $A \in \text{Hom}_G(\pi_1, \pi_2)$  which is also a bijection then we say that  $\pi_1$  and  $\pi_2$  are *isomorphic* unitary representations. An elementary but useful result is the so called Schur's lemma:

**Lemma A.4** (Schur's Lemma) *Let  $\pi : G \rightarrow U(\mathcal{H})$  be an irreducible unitary representation then every intertwining morphism  $A \in \text{Hom}_G(\pi, \pi)$  is a scalar operator, i.e.,  $A = a \cdot \text{Id}_{\mathcal{H}}$  for some  $a \in S^1$ .*

For a proof, see [1, 15].

## A.5 Character Vectors

The final piece of terminology that we will require is the notion of a character vector, which generalizes the notion of an eigenvector.

First, recall the following basic fact from linear algebra:

*Fact 1* A unitary operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  can be diagonalized, which means that there exists an orthogonal decomposition of  $\mathcal{H}$  into a direct sum of eigenspaces

$$\mathcal{H} = \bigoplus_{\lambda \in S^1} \mathcal{H}_{\lambda},$$

where for  $\varphi \in \mathcal{H}_{\lambda}$  we have  $A\varphi = \lambda\varphi$ .

The more general situation occurs when one consider a unitary representation  $\pi : G \rightarrow U(\mathcal{H})$  of a commutative group  $G$ . Such a representation yields a collection  $\{\pi(g) : g \in G\}$  of unitary operators which commute pairwise.

*Fact 2* The unitary operators  $\{\pi(g) : g \in G\}$  can be diagonalized simultaneously, which means that there exists an orthogonal decomposition of  $\mathcal{H}$  into common eigenspaces

$$\mathcal{H} = \bigoplus_{\chi : G \rightarrow S^1} \mathcal{H}_{\chi}.$$

The common eigenspaces are now indexed by functions  $\chi : G \rightarrow S^1$  where each function  $\chi$  encodes the eigenvalues associated with the different operators  $\pi(g)$ ,  $g \in G$ . In more details, for  $\varphi \in \mathcal{H}_{\chi}$ , we have  $\pi(g)\varphi = \chi(g)\varphi$ , for every  $g \in G$ .

It is easy to verify that the functions  $\chi$  which appear in the above decomposition are unitary characters of the group  $G$ , that is,  $\chi(g \cdot h) = \chi(g) \cdot \chi(h)$ , for every  $g, h \in G$ .

The spaces  $\mathcal{H}_{\chi}$  are called a *character spaces* and a vector  $\varphi \in \mathcal{H}_{\chi}$  is called *character vector*.

## A.6 Basic Decompositions

Considering our three basic examples (see Appendix A.3), we obtain, respectively, three orthogonal decompositions of  $\mathcal{H}$  into character spaces.

### A.6.1 Time Shift Invariant Decomposition

For every  $w \in \mathbb{F}_p$ , let  $\psi_w : \mathbb{F}_p \rightarrow S^1$  denote the character  $\psi_w(\tau) = e^{\frac{2\pi i}{p} w \tau}$ . The decomposition into character spaces with respect to the representation  $L$  is

$$\mathcal{H} = \bigoplus_{w \in \mathbb{F}_p} \mathcal{H}_{\psi_w},$$

with  $\dim \mathcal{H}_{\psi_w} = 1$ , for every  $w \in \mathbb{F}_p$ . A function  $\varphi \in \mathcal{H}_{\psi_w}$  if  $\varphi(t) = c \cdot e^{\frac{2\pi i}{p} w t}$ , for some constant  $c \in \mathbb{C}$ .

### A.6.2 Phase Shift Invariant Decomposition

For every  $\tau \in \mathbb{F}_p$ , let  $\psi_\tau : \mathbb{F}_p \rightarrow S^1$  denote the character  $\psi_\tau(w) = e^{\frac{2\pi i}{p} \tau w}$ . The decomposition into character spaces with respect to the representation  $M$  is

$$\mathcal{H} = \bigoplus_{\tau \in \mathbb{F}_p} \mathcal{H}_{\psi_\tau},$$

with  $\dim \mathcal{H}_{\psi_\tau} = 1$ , for every  $\tau \in \mathbb{F}_p$ . A function  $\varphi \in \mathcal{H}_{\psi_\tau}$  if  $\varphi = c \cdot \delta_\tau$ , for some constant  $c \in \mathbb{C}$  and

$$\delta_\tau(t) = \begin{cases} 1, & t = \tau, \\ 0, & t \neq \tau. \end{cases}$$

### A.6.3 Scale Invariant Decomposition

Let us first explain how to describe unitary characters of the multiplicative group  $(\mathbb{F}_p^\times, \cdot)$ . The basic fact that we use is that  $\mathbb{F}_p^\times$  is a cyclic group of order  $p - 1$ , which means that we can write  $\mathbb{F}_p^\times$  in the form  $\{1, r, \dots, r^{p-1}\}$ , for some generator  $r \in \mathbb{F}_p^\times$ . This implies that unitary characters can be specified as follows: for a  $(p - 1)$ th roots of unity,  $\zeta \in \mu_{p-1}$ , let  $\chi_\zeta : \mathbb{F}_p^\times \rightarrow S^1$  be the unitary character given by

$$\chi_\zeta(r^k) = \zeta^k.$$

Now, the decomposition into character spaces with respect to the representation  $S$  is

$$\mathcal{H} = \bigoplus_{\zeta \in \mu_{p-1}} \mathcal{H}_{\chi_\zeta},$$

with  $\dim \mathcal{H}_{\chi_\zeta} = 1$ , for every  $\zeta \neq 1$  and  $\dim \mathcal{H}_{\chi_\zeta} = 2$ , for  $\zeta = 1$ . The one-dimensional space  $\mathcal{H}_{\chi_\zeta}$ ,  $\zeta \neq 1$  is spanned by the function

$$\varphi_\zeta(t) = \begin{cases} \chi_\zeta(t), & t \neq 0, \\ 0, & t = 0, \end{cases}$$

and the two-dimensional space  $\mathcal{H}_{\chi_\zeta}$ ,  $\zeta = 1$  is spanned by the function  $\delta_0$  and the constant function 1.

## Appendix B: Construction of the Oscillator Dictionary

### B.1 Algorithm

We describe an explicit algorithm that generates the oscillator dictionary  $\mathcal{D}_O^s$  associated with the collection of split tori in  $Sp$ .

#### B.1.1 Tori

Consider the standard diagonal torus

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^\times \right\}.$$

Every split torus in  $Sp$  is conjugated to the torus  $A$ , which means that the collection  $\mathcal{T}$  of all split tori in  $Sp$  can be written as

$$\mathcal{T} = \{gAg^{-1} : g \in Sp\}.$$

#### B.1.2 Parametrization

A direct calculation reveals that every torus in  $\mathcal{T}$  can be written as  $gAg^{-1}$  for an element  $g$  of the form

$$g = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix}, \quad b, c \in \mathbb{F}_p. \quad (\text{B.1})$$

If  $b = 0$ , this presentation is unique: In the case  $b \neq 0$ , an element  $\tilde{g}$  represents the same torus as  $g$  if and only if it is of the form

$$\tilde{g} = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix} \begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix}.$$

Let us choose a set of elements of the form (B.1) representing each torus in  $\mathcal{T}$  exactly once and denote this set of representative elements by  $R$ .

#### B.1.3 Generators

The group  $A$  is a cyclic group and we can find a generator  $g_A$  for  $A$ . This task is simple from the computational perspective, since the group  $A$  is finite, consisting of  $p - 1$  elements.

Now, we make the following two observations. First observation is that the oscillator basis  $B_A$  is the basis of eigenfunctions of the operator  $\rho(g_A)$ .

The second observation is, that other bases in the oscillator system  $\mathcal{D}_O^s$  can be obtained from  $B_A$  by applying elements from the set  $R$ . More specifically, for a torus  $T$  of the form  $T = gAg^{-1}$ ,  $g \in R$ , we have

$$B_{gAg^{-1}} = \{\rho(g)\varphi : \varphi \in B_A\}.$$

Concluding, we described the oscillator system

$$\mathcal{D}_O^s = \{\rho(g)\varphi : g \in R, \varphi \in B_A\}.$$



### B.1.4 Formulas

We are left to explain how to write explicit formulas (matrices) for the operators  $\rho(g)$  where  $g \in Sp$ .

First, we recall that the group  $Sp$  admits a Bruhat decomposition  $Sp = B \cup BwB$ , where  $B$  is the Borel subgroup consisting of lower triangular matrices in  $Sp$  and  $w$  denotes the Weyl element

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore, the Borel subgroup  $B$  can be written as a product  $B = AU = UA$ , where  $A$  is the standard diagonal torus and  $U$  is the standard unipotent group

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in \mathbb{F}_p \right\},$$

therefore, we can write the Bruhat decomposition also as  $Sp = UA \cup UAwU$ .

Using the Bruhat decomposition we conclude that every operator  $\rho(g)$ ,  $g \in Sp$ , can be written either in the form  $\rho(g) = M_u \circ S_a$  or in the form  $\rho(g) = M_{u_2} \circ S_a \circ F \circ M_{u_1}$ , where  $M_u$ ,  $S_a$  and  $F$  are the explicit operators which appears in the description of the Weil representation in Sect. 2.3.

*Example B.1* For  $g \in R$ , with  $b \neq 0$ , the Bruhat decomposition of  $g$  is given explicitly by

$$g = \begin{pmatrix} 1 & 0 \\ \frac{1+bc}{b} & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b^{-1} & 1 \end{pmatrix},$$

and consequently

$$\rho(g) = M_{\frac{1+bc}{b}} \circ S_b \circ F \circ M_{b^{-1}}.$$

For  $g \in R$ , with  $b = 0$ , we have

$$g = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

and

$$\rho(g) = M_u.$$

### B.2 Pseudo-code

Below, is given a pseudo-code description of the construction of the *oscillator dictionary*  $\mathcal{D}_O^s$ :

- (1) Choose a prime  $p$ .
- (2) Compute generator  $g_A$  for the standard torus  $A$ .
- (3) Diagonalize  $\rho(g_A)$  and obtain the basis of eigenfunctions  $\mathcal{B}_A$ .
- (4) For every  $g \in R$ :

- (5) Compute the operator  $\rho(g)$  as follows:
  - (a) Calculate the Bruhat decomposition of  $g$ , namely, write  $g$  in the form  $g = u_2 \cdot a \cdot w \cdot u_1$  or  $g = u \cdot a$ .
  - (b) Calculate the operator  $\rho(g)$ , namely, take  $\rho(g) = M_{u_2} \circ S_a \circ F \circ M_{u_1}$  or  $\rho(g) = M_u \circ S_a$ .
- (6) Compute the vectors  $\rho(g)\varphi$ , for every  $\varphi \in B_A$  and obtain the system  $B_{gAg^{-1}}$ .

**Remark B.2** (Running time) It is easy to verify that the time complexity of the algorithm presented above is  $O(p^4 \log p)$ . This is, in fact, an optimal time complexity, since already to specify  $p^3$  vectors, each of length  $p$ , requires  $p^4$  operations.

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