Notes

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1 Section *Homomorphism*

1.1 Mar, 21

Definition:

Suppose there exist two groups $(G, *, I_G)$ and $(H, \circ, I_H)$. A map φ: G is called a *homomorphism* if and only if for any $g_1, g_2 \in G$, $\varphi(g_1 * g_2) = \varphi(g_1) \circ \varphi(g_2)$.

Example: *Sgn homomorphism*: *sgn*: $S_n \to \{\pm 1\}$.

First consider the most simple situation.

(1) When $n = 2$; $D(x_1, x_2) = x_2 - x_1$.

$\sigma \in S_2 = Aut(1, 2)$; σ can act on D, *ie* $\sigma D = D(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}) = x_{\sigma^{-1}(2)} - x_{\sigma^{-1}(1)}$. It can be shown that $\sigma D = D$ or $-D$, Define $sgn(\sigma)$ as $\sigma D = sgn(\sigma) \cdot D$, $sgn(\sigma) = 1$ or -1.

When $\sigma = I$, $\sigma D = D = sgn(\sigma) \cdot D$, $sgn(\sigma) = 1$;

When $\sigma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $\sigma D = -D = sgn(\sigma) \cdot D$, $sgn(\sigma) = -1$;

$$sgn(\sigma) = \begin{cases} 1 & \sigma = I \\ -1 & \sigma \neq I \end{cases}$$

Claim1:

sgn: $S_2 \to \{\pm 1\}$ is a *homomorphism*.

Verify:

	$\sigma = I$	$\sigma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$
$\sigma = I$	I	σ
$\sigma = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$	σ	I

show $sgn(\sigma_1 * \sigma_2) = sgn(\sigma_1) \cdot sgn(\sigma_2)$:

when $\sigma_1 = \sigma_2 = I$, $sgn(I * I) = sgn(I) = 1$;

when $\sigma_1 = \sigma_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $\sigma_1 * \sigma_2 = I$, $sgn(\sigma_1 * \sigma_2) = sgn(I) = 1$;

when $\sigma_1 = I$, $\sigma_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $\sigma_1 * \sigma_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $sgn(\sigma_1 * \sigma_2) = sgn(\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)) = -1$;

when $\sigma_1 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $\sigma_2 = I$, $\sigma_1 * \sigma_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $sgn(\sigma_1 * \sigma_2) = sgn(\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)) = -1$.

$(2) n = 3$; $D(x_1, x_2, x_3) = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1)$; $\sigma \in S_3 = Aut(1, 2, 3)$.

Define: $\sigma D(x_1, x_2, \ldots x_n) = D(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots x_{\sigma^{-1}(n)}) =$
$(x_{\sigma^{-1}(n)} - x_{\sigma^{-1}(n-1)})(x_{\sigma^{-1}(n)} - x_{\sigma^{-1}(n-2)}) \ldots (x_{\sigma^{-1}(2)} - x_{\sigma^{-1}(1)})$.

Claim2:
(a) $\sigma D = sgn(\sigma) D$, $sgn(\sigma) = 1$ or -1;
(b) sgn: $S_n \to \pm 1$ is an *homomorphism*.

Proof of (a):
$sgn(\sigma) = (-1)^k$, $k = \#\{i < j \mid \sigma(i) > \sigma(j)\}$. $(*)$
Example:
if $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$. $sgn(\sigma) = (-1)^1$;
consider σ : *pairs* $(i, j), i < j \to$ *all pairs* $(l, k) \mid l \neq k$
$\sigma(i, j) = (\sigma(i), \sigma(j))$. When $\sigma(i) < \sigma(j)$, there exists $x_{\sigma(i)} - x_{\sigma(j)}$ as a factor in D.
when $\sigma(i) > \sigma(j)$, there exists $-(x_{\sigma(i)} - x_{\sigma(j)})$ as a factor in D. Thus $sgn(\sigma) = (-1)^k$. Then the claim is proved.
QED.

Example: $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$, $sgn = (-1)^2 = 1$.
Proof of (b): $\sigma, \tau \in S_n$, what we need to prove is $sgn(\sigma * \tau) = sgn(\sigma) \cdot sgn(\tau)$.
As the definition of shows, $sgn(\sigma * \tau) D = (\sigma * \tau) D = D((\sigma * \tau)^{-1}(x_1), \ldots (\sigma * \tau)^{-1}(x_n)) = D(\tau^{-1}(\sigma^{-1}(x_1)), \ldots \tau^{-1}(\sigma^{-1}(x_n))) = \sigma(\tau(D)) = \sigma(sgn(\tau) \cdot D) = sgn(\tau) \cdot \sigma D = sgn(\tau) \cdot sgn(\sigma) \cdot D. \Longrightarrow sgn(\sigma * \tau) = sgn(\sigma) \cdot sgn(\tau)$.

1.2 Mar, 23

Theorem:
There exists a unique *onto homomorphism* φ: $S_n \to \{\pm 1\}$.

Proof:
Uniqueness:

Lemma1: $\forall \sigma \in S_n$, $\exists$ a decomposition with unique parity $\sigma = \tau_l * \tau_{l-1} * \ldots * \tau_1$ for some l, where $\tau_i (1 \leq i \leq l)$ are *transposition*.
The parity is determined by the number of τ_i in the decomposition. And the parity is odd or even and doesn't change for different decompositions of σ into product of *transposition*.
Proof:
Step 1:
$\forall \sigma \in S_n$ can be written (uniquely) as a product of *cycles* : $\sigma = c_m * c_{m-1} * \ldots * c_1$.
A *cycle* $c \in S_n$ is a map $\{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$ which is a bijection of the form:
$c = (a_1, a_2, \ldots a_t), 1 \leq a_1 < a_2 \ldots < a_t \leq n$. $(a_1, a_2, \ldots a_t)$ means: $c(a_1) = a_2, c(a_2) = a_3, \ldots c(a_{t-1}) = a_t$.

Step2:
Every *cycle* can be written as a composition of *transposition*. Indeed, $c = (a_1, a_2, \dots a_t) = (a_1, a_2) * (a_2, a_3) * \dots (a_{t-1}, a_t)$.

The uniqueness can be proved following lemma1:
Suppose φ, $\varphi^{'}$ are *onto homomorphism* which map S_n to ± 1.
Step 1: φ and $\varphi^{'}$ send any *transposition* to -1.
(1) All the *transpositions* are conjugate with each other.
It's enough to prove that (a_1, a_2) and (b_1, b_2) are conjugate.
Proof: $a_1, a_2, b_1, b_2 \in \{1, 2, \dots n\}, a_1 \neq a_2, b_1 \neq b_2$.
$(a_2, b_2)(a_1, b_1)(a_1, a_2)(a_1, b_1)(a_2, b_2) = (b_1, b_2)$. Thus (a_1, a_2), (b_1, b_2) are conjugate.
(2) φ and $\varphi^{'}$ send any *transposition* to -1.
Proof: Suppose $\exists$ *transposition* $\tau \in S_n$, $\varphi(\tau) = 1$. Then $\forall$ *transposition* $\sigma \in S_n$, $\exists \gamma \in S_n$, *s.t* $\sigma = \gamma^{-1} \tau \gamma$.
Then $\varphi(\sigma) = \varphi(\gamma^{-1} \tau \gamma) = \varphi(\gamma^{-1}) \varphi(\gamma) \varphi(\tau) \varphi(\sigma^{-1} \sigma) \varphi(\tau) = \varphi(\tau)$. Thus, φ sends any *transposition* to 1. As $\{$*transposition* of $S_n\}$ is a *generator* of S_n, φ sends any element of S_n to 1, which conflicts with φ is an onto mapping. Thus φ and $\varphi^{'}$ sends any *transposition* to -1.
Step 2: φ and $\varphi^{'}$ are onto *homomorphism* which map S_n to ± 1 and satisfy $\varphi(\tau) = -1, \varphi^{'}(\tau) = -1$, $\forall \tau$ that is *transposition* of S_n. Then $\varphi \simeq \varphi^{'}$. Proof: take $\sigma = \tau_l * \dots * \tau_1 \in S_n$, decompose $\varphi(\sigma)$ and $\varphi^{'}(\sigma)$ into:$\varphi(\sigma) = \varphi(\tau_l) \cdot \dots \cdot \varphi(\tau_1) = \varphi^{'}(\tau_l) \cdot \dots \cdot \varphi^{'}(\tau_1) = (-1)^l$. Thus $\varphi \simeq \varphi^{'}$

Existance
Definition of *sgn*: $sgn : S_n \to \pm 1$ is defined by $\sigma D = sgn(\sigma) \cdot D$,
where $D = D(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$,
$\sigma D = D(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$.
Claim 2 has shown that there exists a *homomorphism sgn*, *s.t* $sgn(\sigma) = (-1)^k$,
$k = \#\{(i, j) \mid 1 \leq i < j \leq n, \sigma(j) < \sigma(i)\}$.
In particular, if $\tau = (a, b)$, then $k = 1$ and $sgn(\tau) = (-1)^1 = -1$.

QED.