Introduction to Representation Theory
HW5 - Spring 2016
Due: March 31

1. Let G be a finite group and denote by $K(G)$ its set of conjugacy classes. Consider the character table of size $\#Irr(G) \times \#K(G)$

$$[\ \chi_\pi(C)\ ;\ \pi \in Irr(G) \text{ and } C \in K(G)]\,.$$

   (a) Show that the "rows" of this table are orthonormal with respect to the following inner product on the space of complex valued functions on conjugacy classes

$$\langle f, h\rangle = \frac{1}{\#G} \sum_{C \in K(G)} \#C \cdot f(C)\overline{h(C)}.$$

   (b) Show that the columns of the character table satisfy

      1. $\sum_{\pi \in Irr(G)} \overline{\chi_\pi(C)}\chi_\pi(C) = \#G/\#C.$
      2. $\sum_{\pi \in Irr(G)} \overline{\chi_\pi(C)}\chi_\pi(C') = 0$, for $C \neq C'$.

   Hint: Compute the expansion of the characteristic function 1_C, of the conjugacy class C, as a linear combination of characters

$$1_C = \sum_{\pi \in Irr(G)} a_\pi \chi_\pi.$$

2. In the rest of this document we will prove the following theorem.

Theorem (Burnside). Let G be a finite group of order $p^a q^b$, where p, q are primes and a, b are positive integers. Then G is solvable.

   (a) Recall the definitions of a solvable group and of a simple group. Show that if the theorem is false then there exists a non-abelian simple group G of order $p^a q^b$.

   We want to show that such a simple group does not exists.

   (b) Consider the following theorem

   Proposition 1. Let G be a finite group, and let C be a conjugacy class in G of order p^k where p is a prime and $k > 0$. Then G has a proper nontrivial normal subgroup.

   Show, using the theorem, that simple group G as in Section (a) must have a non-trivial center. This will give a contradiction to its simplicity and we obtain Burnside's theorem. Hint: The class equation (the sum of the cardinalities of conjugacy classes is equal to $\#G$).

(c) The goal now is to prove **Proposition 1.** We will show that it follows from:

Proposition 2. Let (π, V) be an irreducible complex representation of a finite group G and let C be a conjugacy class of G with $gcd(\#C, \dim(V)) = 1$. Then for any $g \in C$, either $\chi_\pi(g) = 0$ or g acts as a scalar on V.

Prove Proposition 1 using the following steps:

1. Show that there exist $g \in C$ such that

$$\sum_{\rho \in Irr(G)} \dim(\rho)\chi_\rho(g) = 0. \tag{1}$$

2. Let G be a finite group. Divide the set $Irr(G)$ to tree subsets
   1. $Irr(G)_0$ - Trivial rep.
   2. $Irr(G)_p$ - Irreps whose dimension is divisible by p.
   3. $Irr(G)_{p'}$ - Non-trivial irreps whose dimension is not divisible by p.

   Prove the following:

   Lemma 1. There exists $\rho \in Irr(G)_{p'}$ with $\chi_\rho(g) \neq 0$ for $1 \neq g \in C$.
   Hint: Note that
$$a = \frac{1}{p} \sum_{\rho \in Irr(G)_p} \dim(\rho)\chi_\rho(g),$$
   is an algebraic integer (i.e., solves polynomial equation with integer coefficients) and use (1). Recall that if $\alpha \in \mathbb{Q}$ is an algebraic integer then $\alpha \in \mathbb{Z}$.

3. Take C as in Proposition 1 and $(\rho, V) \in Irr(G)_{p'}$ as in Lemma 1.
   1. Show that $\rho(g)$ acts on V by scalar for every $g \in C$.
   2. Denote by H the subgroup of G which is generated by ab^{-1}, $a, b \in C$. Show that H is proper normal subgroup of G. Conclude Proposition 1.

(d) We are left to verify Proposition 2. We will use two Lemmas:

Lemma 2. If $z_1, ..., z_n \in \mathbb{C}$ are roots of unity such that $a = \frac{1}{n}(z_1 + ... + z_n)$ is algebraic integer, then $z_1 = ... = z_n$ or $z_1 + ... + z_n = 0$.

1. Prove Lemma 2. Hint: Use the action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on a.

Lemma 3. Let (ρ, V) be a complex be an irreducible complex representation of a finite group G, and let C be a conjugacy class of G. Then for $g \in C$ the the number $\frac{\#C\chi_\rho(g)}{\dim(V)}$ is an algebraic integer.

1. Prove Lemma 3 using the following steps:
   1. Consider the elements $z = \sum_{h \in C} h$ and show that it belongs to the center of the group ring $\mathbb{Z}[G]$. Deduce that $z = \sum_{h \in C} h$ acts on V, via ρ, by some scalar $\lambda \in \mathbb{C}$.
   2. Using the fact that $\mathbb{Z}[G]$ is integral over $\mathbb{Z}$ (i.e., every element in $\mathbb{Z}[G]$ is a root of a polynomial with integer coefficients) show that the above λ is an algebraic integer.
   3. Apply trace on the identity $\lambda = \sum_{h \in C} \rho(h)$ to deduce Lemma 3.

(e) Prove Proposition 2.

Good Luck!