Introduction to Representation Theory
HW4 - Spring 2016
Due: March 15

1. Let V be a finite dimensional vector space over a field $\mathbb{F}$.

   (a) Recall the definition of the trace morphism $Tr : End(V) \to \mathbb{F}$.

   (b) Describe the canonical isomorphism $* : End(V) \to End(V^*)$. Show that $Tr(T) = Tr(T^*)$, where T^* is defined using the morphism $*$.

   (c) Let (π, V), (π^*, V^*) be a representation, and its dual, of a finite group G. Show that

   $$\chi_{\pi^*}(g) = \chi_\pi(g^{-1}), \;\; g \in G.$$

2. Let V be a finite dimensional vector space over $\mathbb{C}$.

   (a) Define the complex conjugate vector space $\overline{V}$ with the same additive structure, but scalar product $\alpha \cdot v := \overline{\alpha} v$.

   (b) Show that $\overline{V}$ and V^* are canonically isomorphic iff V is equipped with an inner product $\langle , \rangle$.

   (c) Let (π, V) be a representation of a finite group G. Define the representation $\overline{\pi}$ of G on $\overline{V}$ and show that $\pi^* \simeq \overline{\pi}$.

3. Let π be a representation of a finite group G.

   (a) Show that there exist a unique idempotent $e_\pi \in \mathcal{A}(G)$, such that

   $$\rho(e_\pi) = \begin{cases} Id_\rho, & \text{if } \rho \simeq \pi; \\ 0, & \text{other.} \end{cases}$$

   (b) Compute the formula of e_π.

4. Let A be a finite Abelian group.

   (a) Consider the inner product on $L(A)$

   $$\langle f, h \rangle = \frac{1}{\#A} \sum_{a \in A} f(a)\overline{h(a)}.$$

   Verify the Plancherel formula $\langle F(f), F(f) \rangle^2 = \#A \langle f, f \rangle$.

   (b) Recall the definition of the Pontryagin dual group $\widehat{A}$.

   (c) Consider the Fourier transform $F : L(A) \to L(\widehat{A})$ given by

   $$F(f)[\psi] = \sum_{a \in A} f(a)\psi(a).$$

   Verify the Fourier inversion formula

   $$f(a) = \sum_{\psi \in \widehat{A}} F(f)[\psi]\psi^{-1}(a).$$

5. Let $\mathbb{F} = \mathbb{F}_q$. Let V be a finite dimensional vector space over V. Describe the dual group $\widehat{V}$ explicitly as follows. Fix a non-trivial additive character $\psi_0 \in \mathbb{F} \to \mathbb{C}^*$. Show that the homomorphism

$$\iota_0 : V^* \to \widehat{V}, \quad \iota_0(\xi)[v] = \psi_0(\xi(v)),$$

is an isomorphism.

6. Consider the natural representation σ of group $\mathbb{F}^* = \mathbb{F}_q^*$ on the space $\mathcal{H} = L(\mathbb{F})$ of complex valued functions on $\mathbb{F} = \mathbb{F}_q$.

- Describe explicitly the decomposition of this space into irreducible components. Namely for any multiplicative character χ of $\mathbb{F}^*$ describe the subspace $\mathcal{H}_\chi \subset \mathcal{H}$ of functions which transform according to χ.
- Show that the Fourier transform F takes $\mathcal{H}_\chi$ into $\mathcal{H}_{\chi^{-1}}$.
- Using the Plancherel formula prove the following Gauss's identity. Fix a non-trivial additive character ψ of $\mathbb{F}$ and a non-trivial multiplicative character χ of $\mathbb{F}^*$. Then the numerical value of

$$G = \sum_{a \in \mathbb{F}^*} \psi(a)\chi(a),$$

satisfies $|G| = \sqrt{q}$.

Good Luck!