Introduction to Representation Theory
HW3 - Spring 2016
Due: March 8

1. *Finite Symplectic Spaces.* A symplectic vector space over a field k is a pair $(V, \langle , \rangle)$, where V is a finite dimensional vector space over k, and $\langle , \rangle$ is a non-degenerate bilinear form $\langle , \rangle$ on V, which is alternating, i.e., $\langle u, v\rangle = -\langle v, u\rangle$ for every $u, v \in V$. Such a form is also called symplectic form.

   For the rest of this HW we fix a symplectic vector space V.

   (a) Define the category of symplectic vector spaces over k and show that every morphism there is injection.

   (b) Show that $\dim(V) = 2n$.

   (c) Show that any two symplectic vector spaces of the same dimension are symplectomorphic, i.e., isomorphic by a symplectic morphism.

   (d) Show that if $I \subset V$ is a subspace on which the symplectic form vanishes then $\dim(I) \leq n$. Such subspace are called isotropic. A maximal isotropic subspace $L \subset V$ is called Lagrangian. Denote by $Lag(V)$ the space of Lagrangians in V.

   (e) Try to compute $\#Lag(V)$ in case $k = \mathbb{F}_q$.

2. *Heisenberg Group.* Consider the set $H = V \times k$ equipped with the product

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\langle v, v'\rangle).$$

   (a) Show that H is a group.

   (b) Compute the center $Z = Z(H)$ of H, and the commutator subgroup $[H, H]$.

   (c) Show that there exists a natural bijection between maximal commutative subgroups of H and Lagrangians in V.

   (d) Show that the symplectic group $Sp(V)$ acts naturally on H by group automorphisms.

   (e) Compute the number of conjugacy classes of H.

3. *Representations of the Heisenberg Group.*

   (a) Construct q^{2n} one dimensional irreducible representations of H.

   (b) Proof the following **Theorem (Stone–von Neumann).** Irreducible representations of H which are non-trivial on Z and agree there are isomorphic. Moreover, for every nontrivial additive character $\psi : Z \to \mathbb{C}^*$ there exists a unique (up to isomorphism) irreducible representation $(\pi_\psi, \mathcal{H}_\psi)$ of H with $\pi_\psi(z)) = \psi(z)Id_{\mathcal{H}_\psi}$.

   (c) Deduce that a representation $\mathcal{H}$ of H with $\dim(\mathcal{H}) > 1$ is irreducible iff it is of dimension q^n.

   (d) Compute the character χ_{π_ψ} for π_ψ as in the theorem.

(e) Let L be a Lagrangian in V. For a fixed $1 \neq \psi : Z \to \mathbb{C}^*$ consider the space

$$\mathcal{H}_{L,\psi} = \{f : H \to \mathbb{C};\ f(l \cdot z \cdot h) = \psi(z) f(h) \text{ for every } l \in L, z \in Z, h \in H\}.$$

Note that the group H acts naturally on $\mathcal{H}_{L,\psi}$ by $[\pi_{L,\psi}(h')f](h) = f(hh')$. Show that $(\pi_{L,\psi}, \mathcal{H}_{L,\psi})$ is irreducible.

(f) Construct a model for each member of $Irr(H)$.

Good Luck!