Math 843 Representation Theory
Spring 2014
HW3 - Due Friday 3/28/14

Remark. Answers for questions, which are not definitions, should be written in the following format:
A) Statement or result.
B) If possible the name of the method you used.
C) The proof or computation.

Representation Theory

1. Prove the following:

   Theorem (Maschke)**.** Let G be a finite group of order n and k a field of characteristic prime to n (i.e. k contains an element $1/n$). Show that any representation of the group G over the field k is completely reducible.

2. Let G be an infinite group and $H \subset G$ a subgroup of finite index. Let (π, G, V) be a complex representation of G and $L \subset V$, a G-invariant subspace. Suppose we know that the subspace L has an H-invariant complement. Show that then it has a G-invariant complement.

3. Let $G = S_n$ denote the group of all permutations of a set I with n elements (symmetric group in n symbols). Consider the natural action of G on the set X of all subsets of I. It has orbits $X_0, X_1, ..., X_n$, where X_i consists of all subsets of size i (note that X_0 has one element). Consider the corresponding representations $\pi_i := (\pi_{X_i}, G, \mathbb{C}(X_i))$ of the group G and decompose them into irreducible components. Describe these components for $i = 0, 1, 2$, and compute their dimensions. Try to do this for the representation $i = 3$. Here we assume that n is large, say $n > 6$.

   Algebras and modules

   Fix a field k. By k-algebra A we mean an associative k-algebra A with 1. We denote by $\mathcal{M} = \mathcal{M}(A)$ the category of (left) A-modules. An A-module M is called <u>simple</u> if it is non-zero and has no submodules different from 0 and M.

4. Let M be an A-module. Consider the space of endomorphisms $\mathcal{E}(M) = End_A(M)$. Show that $\mathcal{E}(M)$ has a structure of k-algebra. Show that if M is simple then $\mathcal{E}(M)$ is a division algebra. Show that if $M = A$ is a free module with one generator then $\mathcal{E}(M)$ is isomorphic to the opposite algebra A^o.

5. Let G be a group. Consider the group algebra $\mathcal{A}(G) = \mathcal{A}(G; k)$. By definition this is a k-vector space with the basis δ_g, $g \in G$, with multiplication given by the formula $\delta_g * \delta_h = \delta_{gh}$. Convince yourself that the category $\mathcal{M}(\mathcal{A}(G))$ is canonically the same as the category $\mathcal{M}(G)$ of representations of the group G over the field k.

   Definition. Let M be an A-module and $L \subset M$ a submodule. We say that L <u>splits</u> as a direct summand in M if there exists a complementary A-submodule $R \subset M$

(this means that $L \subset R = 0$ and $L + R = M$). We say that an A-module M is completely reducible if any submodule in M splits as a direct summand.

6. Show that for a submodule $L \subset M$ the following conditions are equivalent:

   (a) L splits in M as a direct summand.

   (b) Let $\iota : L \to M$ denote the natural imbedding. Then there exists a left inverse morphism of A-modules $p : M \to L$ (this means that $p \circ \iota = Id_L$).

   (c) For any module $N \in \mathcal{M}(A)$ the natural restriction map $res : Hom_A(M, N) \to Hom_A(L, N)$ given by $res(\varphi) = \varphi \circ \iota$ is an epimorphism of sets.

   Remark. The last condition (c) is a typical way how different properties of objects and morphism are characterized in category theory - not in term of inner structure, but in term of relations to other objects.

7. Answer the following:

   (a) Show that if A-module M is completely reducible then any subquotient of M is also completely reducible.

   (b) (*) Show that if A-modules M,N are completely reducible then their direct sum $M \oplus N$ is also completely reducible.

8. Assume that the field k is algebraically closed and A-module M has finite dimension over k. We would like to understand the structure of the algebra $\mathcal{E}(M) = End_A(M)$.

   (a) Show that if M is simple then $\mathcal{E}(M) = k$.

   (b) Show that if M is completely reducible then $\mathcal{E}(M)$ is isomorphic to a finite direct sum of matrix algebras over k.

9. Let A be a finite dimensional algebra over an algebraically closed field k. Let us consider finite dimensional A-modules. Show that the following conditions are equivalent:

   (a) Any A-module is completely reducible.

   (b) The free module A is completely reducible.

   (c) A has a faithful completely reducible module M.

   (d) A is isomorphic to a direct sum of matrix algebras.

Remark You are very much encouraged to work with other students. However, submit your work alone.

Good Luck!