Introduction to Representation Theory
HW2 - Spring 2016
Due March 1

1. *Representations of Abelian groups.*

   (a) Show that irreps of Abelian groups are one dimensional.

   (b) Let A be a finite Abelian group. Denote by $\widehat{A}$ the Pontryagin dual of all additive characters
   $$\widehat{A} = \{\psi : A \to \mathbb{C}^*;\ \psi(a+a') = \psi(a)\psi(a'),\ \forall a, a' \in A\}.$$
   Show that the canonical isomorphism
   $$L(A) \to \bigoplus_{\pi \in Irr(A)} End(V_\pi),$$
   induce a linear map
   $$F : L(A) \to L(\widehat{A}),$$
   called the Fourier transform.

   (c) Write an explicit formula for F above in the case of $A = \mathbb{Z}/n\mathbb{Z}$.

2. *Fourier Transform.* For a group G consider the natural transformation
   $$\Pi : \mathcal{A}(G) \to \bigoplus_{\pi \in Irr(G)} End(V_\pi).$$
   Consider the linear map
   $$\Psi : \bigoplus_{\pi \in Irr(G)} End(V_\pi) \to \mathcal{A}(G),$$
   induced by $\Psi[A](g) = \frac{\dim(\pi)}{\#G} Tr(\pi(g)A)$ for $\pi \in Irr(G)$ and $A \in End(V_\pi)$. Show directly that $\Pi \circ \Psi = I$ and $\Psi \circ \Pi = I$.

3. *Dual representation.* Let (π, V) be a finite dimensional representation of a (finite) group G.

   (a) Define the dual representation π^* of G on V^*.

   (b) Show that $\pi \simeq \pi^*$ if and only if there exists a non-degenerate invariant bilinear form on G.

   (c) Show that if π is irreducible then, up to scalar multiple, there exists at most one non-degenerate invariant bilinear form on V.

4. Let V and W be two finite dimensional vector spaces over a field $\mathbb{F}$. Recall the two constructions of tensor product of V and W. In each case write down a natural basis for the tensor product in term of basis for V and basis for W.

5. For operators T on V and S on W define the operator $T \otimes S$ on $V \otimes W$ and show that $Tr(T \otimes S) = Tr(T)Tr(S)$.

Good Luck!