Math 842 - Applied Algebra, Spring 2015
HW2 - Random Walk on $G = S_n$ using Transpositions
Due: Thursday 5/7/15

Let $G = S_n$ and $S \subset G$ the set that includes all transpositions and the identity permutation id. Denote by $\mathcal{H}$ the space $F(G)$ of complex valued functions on G.

1. Define carefully the distributions P_S, P_G, P_S^k, $k \geq 1$.
2. For two complex valued functions $f_1, f_2 \in \mathcal{H}$ we define their convolution by
$$(f_1 * f_2)(\sigma) = \sum_{\sigma_2 \cdot \sigma_2 = \sigma} f_1(\sigma_1) f_2(\sigma_2).$$
   (a) For $k \geq 1$, and f a function on G, define the k-th convolution f^{*k} and write an explicit formula for it.
   (b) Explain why $P_S^k = P_S^{*k}$ for every $k \geq 1$.
3. Explain why for every $\sigma, \lambda \in G$ we have $P_S(\lambda\sigma\lambda^{-1}) = P_S(\sigma)$.
4. Define the L_1 distance on the space $\mathcal{H}$ by
$$d(f_1, f_2) =?$$
5. We say that two elements σ,$\rho \in G$ are conjugate if there exists $\lambda \in G$ such that $\lambda\sigma\lambda^{-1} = \rho$. The conjugacy class of an element $\sigma \in G$ is the subset $C_\sigma \subset G$ of all elements which are conjugate to σ.
   (a) Show that $C_\sigma = C_{\lambda\sigma\lambda^{-1}}$ for every $\sigma, \lambda \in S_n$.
   (b) Denote by G/G the set whose elements are the conjugacy classes of G.
   (c) For $G = S_3$ compute explicitly G/G.
   (d) Show that if $C \neq D$ are conjugacy classes in G then $C \cap D = \varnothing$ the empty set.
   (e) Show that
$$G = \bigcup_{C \in G/G} C,$$
   where the (disjoint) union is taken over all conjugacy classes in G.
6. Define what is a partition $\pi(n)$ of n. Denote by $\mathcal{P}(n)$ the set of partitions of n.
   (a) Define for $G = S_n$ the bijection
$$\nu : G/G \to \mathcal{P}(n).$$
   Do it using the following steps:
      1. Explain how starting with a permutation $\sigma \in S_n$ one get, using decomposition into composition to disjoint cycles, a partition.

2. Show that $\sigma \in S_n$ is conjugate to $\rho \in S_n$ then the partitions you obtained from σ and ρ, respectively, are the same.
3. Define the map ν.

(b) Describe explicitly the bijection ν for $G = S_3$.

(c) For a conjugacy class $C_{\pi(n)}$ in S_n which is associated with a partition $\pi(n) \in \mathcal{P}(n)$ compute explicitly the cardinality $\#C_{\pi(n)}$.

7. Show that for a conjugacy class $C \subset G$ there is a well defined value $P_S^k(C)$.

8. Prove the identity

$$d(P_S^k, P_G) = \sum_{C \in G/G} \left| \#C \cdot P_S^k(C) - \frac{\#C}{\#G} \right|.$$

9. Explain the method described in class to compute approximately all the quantities $\#C \cdot P_S^k(C)$, where C a conjugacy classes in G.

10. Using 8. and 9. above, for each of $n = 12, 16, 20$ and $k = 1, ..., n$ plot the graph (as a function of k) of $d(P_S^k, P_G)$.

11. Based on your numerics 'guess' the cut-off for random walk on S_n using transpositions. Namely, for a fixed n what k is enough so that the distance $d(P_S^k, P_G)$ is considerably small.

Good Luck!