Math 842 Spring 2011
Homework#3, 03/02/11— The General Linear Group, Representation of a Group

Remark. Answers should be written in the following form at:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. *The group* $GL(n, \mathbb{C})$. Consider the set of all invertible $n \times n$ complex matrices $GL(n, \mathbb{C}) = \{A \in Mat(n, \mathbb{C}); \text{There exists } A^{-1}\}$. Note that $GL(n, \mathbb{C})$ is in a natural way identifies with the collection of all linear isomorphisms from $\mathbb{C}^n$ to itself. Show that with the operation $\cdot$ of multiplication of matrices, and I, the identity matrix, as the identity element, the triple $(GL(n, \mathbb{C}), \cdot, I)$ is a group. It is called the complex *general linear group* of order n.

2. *The group* $GL(V)$. Let V be a complex vector space of dimension n. Show that the triple $(GL(V), \circ, I)$, where $GL(V) = \{T : V \to V; T \text{ is linear and } \exists\ T^{-1}\}$, $\circ$ is composition of operators, and $I : V \to V$ denotes the identity transformation, is a group. It is called the *general linear group* of V.

3. Let V be a complex vector space of dimension n. Choose a basis $B = \{v_1, ..., v_n\}$ for V. For a linear operator $T : V \to V$, consider the matrix $[T]_B \in Mat(n, \mathbb{C})$ that represent T with respect to B (see HW2).

   (a) Show that if $T \in GL(V)$ then $[T]_B \in GL(n, \mathbb{C})$ i.e., it is invertible.

   (b) Let $(G, \cdot, 1_G)$ and $(G', *, 1_{G'})$ be two groups. A map $\varphi : G \to G'$ is called *isomorphism* if it is homomorphism, i.e., satisfies $\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2)$ for every $g_1, g_2 \in G$, and one-to-one, and onto.

      1. Show that the map $[]_B : GL(V) \to GL(n, \mathbb{C})$ is an homorphism (use HW2).
      2. Show that $[]_B : GL(V) \to GL(n, \mathbb{C})$ is one-to-one by showing that $\ker([]_B) = \{I\}$.
      3. Show that $[]_B : GL(V) \to GL(n, \mathbb{C})$ is onto.
      4. Deduce that $[]_B : GL(V) \to GL(n, \mathbb{C})$ is an isomorphism of groups.

4. *Representation of a group.* Let G be a finite group. A *representation* of G on a finite-dimension complex vector space V is an homomorphism $\rho : G \to GL(V)$. We will usually denote a representation by a triple (ρ, G, V). Choose a basis B for V and consider the isomorphism $[]_B : GL(V) \to GL(n, \mathbb{C})$. Show that the composition $[]_B \circ \rho : G \to GL(n, \mathbb{C}) = GL(\mathbb{C}^n)$, $g \mapsto [\rho(g)]_B$, is a representation of G on $\mathbb{C}^n$.

Good Luck!