Theoretical

Linear transformations preserving an inner product

Consider the three dimensional vector space

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

consisting of all triples of real numbers. We will use $\langle\ ,\ \rangle$ to denote the usual inner product,

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\rangle = x_1x_2 + y_1y_2 + z_1z_2.$$

The orthogonal group, denoted O(3), is the set of all linear transformations (given by 3×3 matrices) that preserve this inner product, i.e.,

$$\mathrm{O}(3) = \{A : \mathbb{R}^3 \to \mathbb{R}^3 \mid \langle Au, Av\rangle = \langle u, v\rangle \text{ for all } u, v \in \mathbb{R}^3\}.$$

1. Show that:
   (a) If A and B is in O(3), $A \cdot B$ is in O(3), where $\cdot$ denotes matrix multiplication.
   (b) The 3×3 identity matrix I_3 is in O(3).
   (c) Observe that:
      i. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$, i.e., matrix multiplication is associative,
      ii. The identity matrix I_3 satisfies $AI_3 = A$ for every $A \in \mathrm{O}(3)$,

      Show that:

      iii. If $A \in \mathrm{O}(3)$, then A is invertible and A^{-1} is also in O(3).

   Overall, parts (a) - (c) show that $(\mathrm{O}(3), \cdot, I_3)$ is a group.

The special orthogonal group, denoted SO(3), is the set of all orthogonal transformations that have determinant 1, i.e.,

$$\mathrm{SO}(3) = \{A \in \mathrm{O}(3) \mid \det A = 1\}.$$

Such matrices are called rotations, and SO(3) is also called the rotation group of $\mathbb{R}^3$.

2. Show that $(\mathrm{SO}(3), \cdot, I_3)$ is a group, where $\cdot$ denotes matrix multiplication, and I_3 is the 3×3 identity matrix.
3. Suppose that $A \in \mathrm{SO}(3)$. Show that there is a vector $v \in \mathbb{R}^3$ such that $Av = v$.

Orthogonal transformations on $\mathbb{R}^2$

Consider the two dimensional vector space

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}.$$

4. Define the standard inner product on $\mathbb{R}^2$.
5. Define the orthogonal group O(2) and the special orthogonal group SO(2).
6. Given $\theta \in [0, 2\pi]$, we define the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

   This matrix rotates vectors in $\mathbb{R}^2$ counter-clockwise by an angle of θ. Show that R_θ is in SO(2).

   Remark: In fact, it can be shown that <u>every</u> element of SO(2) is R_θ for some θ.

Symmetries of an object in space

The unit cube centered at the origin in $\mathbb{R}^3$ is denoted $\mathcal{O}$. The group of symmetries of the cube, denoted O, is the set of all rotations that preserve the cube, i.e.,

$$O = \{A \in \mathrm{SO}(3) \mid A(\mathcal{O}) = \mathcal{O}\}.$$

7. Show that $(O, \cdot, I_3)$ is a group, where $\cdot$ denotes matrix multiplication, and I_3 is the 3×3 identity matrix.

Symmetries of a square

The square centered at the origin in $\mathbb{R}^2$ is denoted $\mathcal{S}$. The group of symmetries of the square, denoted S, is the set of all orthogonal transformations that preserve the cube.

$$S = \{A \in \mathrm{O}(2) \mid A(\mathcal{S}) = \mathcal{S}\}.$$

8. Show that $\#S = 8$, i.e., the number of elements in S is 8.

9. Explicitly compute all the matrices in S.

Permutations

Let Y be a non-empty set. A one-to-one and onto function from Y to itself is called a permutation. The collection of all permutations of Y, denoted S_Y, is called the permutation group of Y.

10. Show that $(S_Y, \circ, \mathrm{id})$ is a group, where $\circ$ denotes composition of functions, and $\mathrm{id} : Y \to Y$ is the identity permutation, i.e., $\mathrm{id}(y) = y$ for all $y \in Y$.

Group Actions & Stabilizers

Suppose G is a group with operation $*_G$ and identity element 1_G, and X is a set on which G acts. Recall that this means that for every element $g \in G$ and every point $x \in X$ there is a unique element $g \cdot x$ in X, and this assignment satisfies

- Associativity: $g \cdot (h \cdot x) = (g *_G h) \cdot x$,
- Identity: $1_G \cdot x = x$ for all $x \in X$,

for all $g, h \in G$ and all $x \in X$. In this case we also say that X is a G-set. If Y is a subset of X, we can consider the set

$$g \cdot Y = \{g \cdot y \mid g \in G, y \in Y\},$$

i.e., the set obtained by transforming the elements of Y using g. The stabilizer of Y in G, denoted $\mathrm{Stab}_G(Y)$, is the collection

$$\mathrm{Stab}_G(Y) = \{g \in G \mid g \cdot Y = Y\},$$

i.e., the elements of G that fix Y.

11. Show that $(\mathrm{Stab}_G(Y), *_G, 1_G)$ is a group.

    Remark: This is an alternative way to conclude that O is a group (see Problem 7).

Numerics

We've prepared an image of a square as a `.mat` file, which is MATLAB's standard file format. SciPy includes routines for easily loading and saving `.mat` files.

For this assignment, please hand in (for example) a Python program, e.g., a `.py` file, that completes the following tasks. Use Python comments to answer questions that aren't explicit computations. Make sure your program runs!

Loading and manipulating data

1. Load `square.mat` and bind a variable named `square` to this data. See the `loadmat` function in the module `scipy.io`.

All variables in Python are of a particular type, e.g., integer, floating point numbers, etc. `square` is a matrix representing a square: it is an $N \times N$ matrix that is mostly 0's, and has non-zero entries in a square around the center of the matrix.

To get the shape of a matrix in numpy, access the matrix's `.shape` property. Matrices and vectors in Python are indexed starting at 0.

2. Print the shape of `square` and the matrix coordinates of the center of the square.

In Python, the i, j entry of a matrix is accessed by `square[i,j]`.

3. Print the value of `square` at the center of the image.

You can access the ith row of a matrix by `square[i,:]`, and the jth column by `square[:,j]`, i.e., we think of `:` as iterating over that coordinate. You can also specify ranges, e.g. `square[0:3,:]` will display the first 3 rows of the matrix.

4. Print the 25th row of `square`. What is the meaning of this data?

Displaying data

SciPy provides convenient plotting and graphing capabilities for Python.

5. Display `square` as an image. See the `imshow` function in the `scipy.misc` module.

If we interpret `square` as sampled data about the origin in $\mathbb{R}^2$, we can apply a geometric rotation to the figure.

6. Display `square` rotated by 90 degrees counter clockwise. See `rot90` in `numpy`.

Notice that the corners of the square are colored.

7. Compare the rotated and unrotated versions of the square. What permutation of the vertices does this rotation correspond to?

Eigenvalues of matrices

In these exercises we'll examine the eigenvalues of rotation matrices. Recall that an element of SO(3) can be represented as a 3×3 matrix A such that, interpreted as vectors, each column of A is unit-length, the inner product of each pair of columns is 0, and $\det(A) = +1$.

8. Generate a random rotation matrix. There are several ways to do this. Here's one idea:
   (a) Set c to be a random vector in the $[-1, 1]^3$ cube. See the `uniform` class in the `scipy.stats` module.
   (b) Normalize c to obtain a vector on the unit sphere. See `norm` in `numpy.linalg`.
   (c) Set b to be the cross product of c with another random vector on the unit sphere. See `cross` in `numpy`.
   (d) Normalize b. We know have a pair of randomly chosen vectors that are perpendicular.
   (e) Set $a = \text{cross}(b, c)$.
   (f) We now have three randomly chosen orthonormal vectors. Set $A = [a, b, c]$.
9. Print the eigenvalues of A. See `eigvals` in `numpy.linalg`.

10. What can you say about the relationship between the eigenvalues of A? Recall that in Problem 1 of the theoretical portion, you showed that any $A \in \mathrm{SO}(3)$ has 1 as an eigenvalue.
11. Generate 500 random rotation matrices, and compute their eigenvalues.
12. Plot the arguments (polar angle) of the eigenvalues you've obtained. What does the distribution look like?