Math 842 Spring 2011
Homework#1, 02/02/11— Groups, Digital Signals, Heisenberg Operations

Remark. Answers should be written in the following form at:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. *Group theory.* Let $(G, \circ, 1)$ be a group.

   (a) Uniqueness of identity. Suppose that $1'$ satisfies $1' \cdot g = g \cdot 1' = g$ for every $g \in G$. Show that $1' = 1$.

   (b) Uniqueness of inverses. Suppose $g, g', g'' \in G$ with $g \cdot g' = g' \cdot g = 1$ and $g \cdot g'' = g'' \cdot g = 1$. Show that $g' = g''$.

   (c) Show that for every $g, h \in G$ we have $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$.

2. *The space of digital signals.* Consider the Hilbert space $\mathcal{H} = \mathbb{C}(\mathbb{Z}/N)$ of complex valued functions on $\mathbb{Z}/N = \{0, 1, ..., N-1\}$. Recall that the inner product is given by

$$\langle f_1, f_2 \rangle = \sum_{t \in \mathbb{Z}/N} f_1(t)\overline{f_2(t)},$$

   with $\overline{a}$ denotes the complex conjugate of the complex number a.

   (a) Show that the basis $B = \{\delta_x;\ x \in \mathbb{Z}/N\}$ is an orthonormal basis for $\mathcal{H}$, i.e.,

$$\langle \delta_x, \delta_y \rangle = \delta_{x=y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

   (b) Suppose V is a vector space over $\mathbb{C}$ of dimension n, and that $B = \{v_1, ..., v_n\} \subset V$ is an orthonormal collection of vectors. i.e.,

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

   Show that B is a basis for V.

   (c) Show that $\sum_{0 \leq t \leq N-1} e^{\frac{2\pi i}{N} t} = 0$.

   (d) Show that $B = \{\psi_\omega(t) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} \omega \cdot t};\ \omega = 0, 1, , , , , N-1\}$ is an orthonormal basis for the space of digital signals. Hint: Use 2.b. and 2.c. above.

3. *Heisenberg operations.*

(a) Let V and W be two vector spaces over $\mathbb{C}$. We say that V and W are *isomorphic,* denoted $V \simeq W$, if there is an invertible linear transformation $\alpha : V \to W$. Show that the natural map

$$\alpha : \mathbb{C}(\mathbb{Z}/N) \to \mathbb{C}^N, \tag{1}$$

which sends

$$\delta_x \longmapsto \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \text{the vector with one in the } x\text{'th coordinate,}$$

is an isomorphism.

(b) We would like calculate the matrices representing the Heisenberg operators. It is not hard to verify that if $O : \mathbb{C}(\mathbb{Z}/N) \to \mathbb{C}(\mathbb{Z}/N)$ is a linear operator then the corresponding operator on $\mathbb{C}^N$, under the identification (1), is given by the $N \times N$ matrix $[O]$ with entries $[O](x, y) = \langle O\delta_y, \delta_x \rangle$, $0 \le x, y \le N - 1$.

1. Show that the matrix $[L_\tau]$, L_τ the time-shift operator, is given by $[L_\tau](x, y) = \delta_{y=x+\tau}$ for every $y, x, \tau \in \mathbb{Z}/N$. Here, $\delta_{y=x+\tau}$ is the function on $\mathbb{Z}/N \times \mathbb{Z}/N$ given by
$$\delta_{y=x+\tau} = \begin{cases} 1 & \text{if } y = x + \tau, \\ 0 & \text{otherwise.} \end{cases}$$
2. Show that the matrix $[M_\omega]$, M_ω is the phase-shift operator, is given by the diagonal matrix
$$[M_\omega](x, y) = e^{\frac{2\pi i}{N}\omega x}\delta_{y=x}$$
for every $y, x, \omega \in \mathbb{Z}/N$.
3. Show that the matrix $[\pi(\tau, \omega, z)]$ of the Heisenberg operator $\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\tau\omega+z\}} \cdot M_\omega \circ L_\tau$, where $\circ$ denotes composition of linear operators, is equal to
$$[\pi(\tau, \omega, z)](x, y) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\tau\omega+z\}} \cdot e^{\frac{2\pi i}{N}\omega x} \cdot \delta_{y=x+\tau}.$$

Good Luck!