Math 741 - Fall 2015

Homework 9. Presentations Mon Dec. 7. 2015 5pm.

1. **Tensor Product.** Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$.

   a. Recall the definition of the pair $(V \otimes W, \otimes)$ which we call the <u>tensor product</u> of V and W.

   b. Let $W^\star$ denote the dual space of W, and consider the vector space $Hom(W^\star, V)$ of all linear maps from $W^\star$ to V. We define $hom : V \times W \to Hom(W^\star, V)$ to be the bilinear map $(v, w) \mapsto hom(v, w)$ where $hom(v, w)(w^\star) = w^\star(w) \cdot v$ for every $v \in V$, $w \in W$ and $w^\star \in W^\star$. Show that the pair $(Hom(W^\star, V), hom)$ also satisfies the axiom of the tensor product of V and W.

   c. Conclude that $dim(V \otimes W) = dim(V) \cdot dim(W)$.

2. Let G be a finite group and (π, V) an irreducible representation of G. Show that $dim(V)$ is finite.

3. **Irreducible representations of Abelian groups.**

   a. Let A be an abelian group (not necessarily finite). Show that every complex finite-dimensional, irreducible representation of A is one-dimensional.

   b. Let $G = \mathbb{Z}$, $V = \mathbb{C}^2$, and define $\pi : G \to GL(V)$ by

$$\pi(n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n .$$

   This gives a representation (π, V) of G. Find a G-invariant subspace $L \subset V$ which has no G-invariant complement $L^0 \subset V$ (we say $W, U \subset V$ are complements of each other if $W \oplus U = V$).

4. **Intertwining numbers and permutation representations.** Let (π, V) and (ρ, W) be two finite dimensional representations of a finite group G. The integer $< \pi, \rho >= dim(Hom(\pi, \rho))$ is called the <u>intertwining number</u> of π and ρ.

   a. Let π, ρ, τ be representations of G.

i. Show that $< \pi \oplus \tau, \rho >=< \pi, \rho > + < \tau, \rho >$.

ii. Show that the intertwining number is symmetric, i.e., $< \pi, \rho >=< \rho, \pi >$.

iii. Suppose π, ρ are irreducible complex representations. Show that we have $< \pi, \rho >= 1$ if π and ρ are isomorphic, and $< \pi, \rho >= 0$ otherwise.

Now, let X and Y be finite G-sets, and denote by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ the sets of complex valued functions on X and Y respectively. Consider the representation $(\pi_X, \mathbb{C}(X))$ given by

$$[\pi_X(g)f](x) = f(g^{-1}x).$$

This is called the permutation representation of G on $\mathbb{C}(X)$, and in a similar manner we obtain the permutation representation of G on $\mathbb{C}(Y)$. Note that the actions on X and Y give rise to an action on $X \times Y$ given by $g \cdot (x, y) = (gx, gy)$.

b. Show that $< \pi_X, \pi_Y >= \#(X \times Y)/G$, where $\#(X \times Y)/G$ denotes the number of orbits of $X \times Y$ under the action of G.

5. Let $X = \{1, \dots, n\}$ and let $G = S_n$. We consider X as a G-set under the natural action. This gives rise to the permutation representation of G on X defined in the previous problem.

a. Show that $< \pi_X, \pi_X >= 2$ (where $< \pi_X, \pi_X >$ is the interwining number defined in problem 4).

b. Let $\mathcal{H} = \mathbb{C}(X)$ and define the subspaces

$$\mathcal{H}_0 = \{f \in \mathcal{H} | \sum_{x \in X} f(x) = 0\},$$

$$\mathcal{H}_c = \{f \in \mathcal{H} | \ f \text{ is constant}\}.$$

Using part 5.a. show that $\mathcal{H}_0$ is irreducible, and that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_c$ is a decomposition of $\mathcal{H}$ into a direct sum of irreducible representations of G.

Good Luck!