Math 741 - Fall 2015

Homework 8. Presentations: Wed Dec. 2. 2015 5pm.

1. **Solvable Groups.** Show that the following are equivalent for a group G.

   a. G has a solvable series.

   b. We have $G \in \mathcal{S}$. Here $\mathcal{S}$ is the minimal collection of all groups such that,

   i. Every abelian group is in $\mathcal{S}$.

   ii. If $1 \to H' \to H \to H'' \to 1$ is a short exact sequence and $H', H'' \in \mathcal{S}$ then $H \in \mathcal{S}$.

2. **Flags.** Let V be a n-dimensional vector space over a field k.

   a. Define the notion of a full flag in V.

   b. Define the natural action of $G = GL(V)$ on the set $\mathcal{F}$ of all flags in V. Show that this action is transitive.

   c. Specialize to the case $k = \mathbb{F}_q$. Compute $\#\mathcal{F}$ for the following cases.

   i. dim(V)=2.

   ii. dim(V)=3.

   iii. dim(V)=n.

3. **Borel Subgroups.** Let V be a n-dimensional vector space over a field k. Fix a full flag

$$F : 0 = V_0 \subset V_1 \subset \ldots V_{n-1} \subset V_n = V$$

in V and consider the following subgroups of $GL(V)$:

$$B_F = Stab_{GL(V)}(F),\ U_F = \{u \in B_F |\ u \text{ acts on } V_{i+1}/V_i \text{ trivially }\},\ T_F = (k^*)^n.$$

   a. Show that B_F fits into a short exact sequence $1 \to U_F \to B_F \to T_F \to 1$.

   b. Show that U_F is solvable. (For example, show that the series for U_F that we defined in class is a solvable series).

   c. Deduce that B_F is solvable.

4. **Centralizers.** Consider the element

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G = SL_2(\mathbb{F}_q).$$

Show that the group $Cent_G(w) = \{g \in G)|gw = wg\}$ is Abelian and compute its order.

5. **Fourier Transform.** Let $\mathcal{H} = \mathbb{C}(\mathbb{F}_q)$ for q a power of an odd prime p, and equip $\mathcal{H}$ with the standard inner product $<,>$. Consider the Fourier transform $F : \mathcal{H} \to \mathcal{H}$ defined by,

$$F[f](y) = p^{-1/2} \sum_{x \in \mathbb{F}_q} \psi(-yx) f(x),$$

where $\psi \neq 1$ is an additive character of $\mathbb{F}_q$.

a. Show that F is unitary. That is $< f_1, f_2 >=< F[f_1], F[f_2] >$ for all $f_1, f_2 \in \mathcal{H}$.

b. Show that $F^2[f](x) = f(-x)$ for all $f \in \mathcal{H}$.

c. Define $G = \sum_{x \in \mathbb{F}_p} \psi(x^2)$ and consider the Legendre symbol $\sigma : \mathbb{F}_p \to \mathbb{C}$ defined by,

$$\sigma(x) = (\frac{x}{p}) = \begin{cases} 1 \text{ if x is a square.} \\ 0 \text{ if } x = 0. \\ -1 \text{ if x is not a square.} \end{cases}$$

i. Show that $G = \sum_{x \in \mathbb{F}_p} \psi(x)\sigma(x)$ (this is called Gauss's sum identity).

ii. Show that $G = p^{1/2} F^{-1}[\sigma](1)$.

iii. Conclude that $G^2 = (\frac{-1}{p}) \cdot p$.

6. **Fourier Transform is an Intertwiner.** Consider the Heisenberg group, $H = V \times \mathbb{F}_q$ with q a power of an odd prime p. Here $V = \mathbb{F}_q \times \mathbb{F}_q$, and the group operation is $(v, z) \bullet (v', z') = (v + v', z + z' + 2^{-1} w(v, v'))$ where $w(v, v') = \det \begin{pmatrix} v \\ v' \end{pmatrix}$. Let $\psi \neq 1$ be an additive character. We define two representations π_X, π_Y of H on $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$ as follows.

$$[\pi_X(x, y, z)f](t) = \psi(2^{-1}xy + z)\psi(yt)f(t + x),$$

$$[\pi_Y(x, y, z)g](s) = \psi(-2^{-1}xy + z)\psi(xs)g(s - y),$$

for every $x, y, z \in \mathbb{F}_q$ and every $f \in \mathcal{H}$. Show that $F : \mathcal{H} \to \mathcal{H}$ is an intertwiner between π_X and π_Y.

Good Luck!