Math 741 - Fall 2015

Homework 7. Presentations: Mon Nov. 23. 2015 5pm.

1. **Equivalent definitions for Nilpotent group of class n.** Show that the following are equivalent for a group G.

   i. There exists $m \geq 0$ such that $Z^m(G) = G$, and the smallest such m is n. Here $Z^0(G) = 1$ and $Z^{i+1}(G) = \{g \in G : \ [g, x] \in Z^i(G) \text{ for all } x \in G\}$.

   ii. There exists a central series for G. That is, a sequence of normal subgroups, $1 = G_0 < G_1 < \ldots < G_m = G$, such that $G_{i+1}/G_i \subset Z(G/G_i)$ for all i. Moreover, the shortest such series has length n.

   iii. We have $G \in \mathcal{N}$. Here $\mathcal{N}$ denotes the minimal collection of groups with the following properties:

   1. $A \in \mathcal{N}$ for all Abelian groups A.

   2. If $1 \to G' \to G \to G'' \to 1$ is a central extension, and $G', G'' \in \mathcal{N}$, then $G \in \mathcal{N}$. Moreover, n is the smallest number of steps with which G can be obtained from an Abelian group using iterations in iii.2. (For nontrivial Abelian groups n=1, so to obtain G we start at step 1 with an Abelian group).

2. **Jordan-Holder series for $\boldsymbol{S_4}$.** Find a Jordan-Holder series for S_4. In particular, give the composition factors.

3. **Jordan-Holder series need not be unique.** Write two Jordan-Holder series for C_{12}, $\{G_i\}$ and $\{G'_i\}$, for which there is no permutation σ such that G_i is isomorphic to $G'_{\sigma(i)}$ for every i. However, note that the composition factors are the same (isomorphic) after a suitable permutation of the indices.

4. **Jordan-Holder series for the standard Unipotent group.** Let $U_n(\mathbb{F}_q)$ be the group of $n \times n$ upper triangular matrices over the finite field $\mathbb{F}_q$ with 1 on the diagonal. In each of the following cases, give a composition series and compute the associated composition factors.

   a. n=3.

b. n=4.

c. General n.

5. **Some solvable subgroups of GL_2.** Show that the following groups are solvable.

   a. The subgroup $B < GL_2(\mathbb{F})$ consisting of upper triangular matrices.

   b. $B = Stab_{GL(V)}(L)$ where V is a 2-dimensional vector space over a field $\mathbb{F}$, and L is a line in V (that is, L is 1-dimensional subspace of V).

6. **Borel subgroups are solvable.** Let V be a 3-dimensional vector space over a field $\mathbb{F}$. A flag in V is a sequence of subspaces

$$F : 0 \subset L \subset P \subset V,$$

   where L is a line (i.e. a 1-dimensional subspace of V) and P is a plane (i.e. a 2-dimensional subspace of V).

   a. Describe the natural action of $GL(V)$ on the set $\mathcal{F}$ of all flags in V.

   b. Fix some $F \in \mathcal{F}$ and consider the group $B_F = Stab_{GL(V)}(F)$. Show that B_F is solvable.

Good Luck!