Math 741 - Fall 2015

Homework 6. Presentations: Mon Nov. 16. 2015 5pm.

In problems 1 and 2, you will find an alternative proof of the first Sylow Theorem.

1. Fix $n, p \in \mathbb{N}$ with p prime. Let $U_n(\mathbb{F}_p)$ denote the subgroup of $GL_n(\mathbb{F}_p)$ consisting of all upper triangular matrices with 1 on the diagonal. Show that $U_n(\mathbb{F}_p)$ is a p-Sylow subgroup of $GL_n(\mathbb{F}_p)$.

2. Let H be a finite group with $|H| = n$ and $p|n$.

   a. An embedding is an injective homomorphism. Construct an embedding

   $$\iota : H \to GL_n(\mathbb{F}_p).$$

   b. Conclude that H has a p-Sylow subgroup. (Hint: Use the 2nd Sylow theorem from class).

3. **Semidirect product of Abelian groups need not be Abelian.** Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$ or 5, and define

   $$B = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}^*, b \in \mathbb{F} \}.$$

   a. Show that B is not Abelian.

   b. Show that B fits into a split short exact sequence

   $$\begin{tikzcd} 1 \arrow[r] & \mathbb{F} \arrow[r, "\iota"] & B \arrow[r, "pr"] & \mathbb{F}^\star \arrow[l, bend left, "s"] \arrow[r] & 1 \end{tikzcd} .$$

4. **Unipotent matrices.**

   a. Let $\mathbb{F}$ be a field, and let $U_4(\mathbb{F})$ be the group of all 4×4 upper triangular matrices with 1 on the diagonal. Show that $U_4(\mathbb{F})$ is nilpotent and compute its nilpotency class.

   b. More generally, let $U_n(\mathbb{F})$ be the group of all $n \times n$ upper triangular matrices with 1 on the diagonal. Show that $U_n(\mathbb{F})$ is nilpotent and compute its nilpotency class.

5. **Upper triangular matrices.** Let $\mathbb{F}$ be a field, and let $B < GL_2(\mathbb{F})$ be the group of upper triangular matrices. Show that B is not nilpotent.

6. **Equivalent notions of nilpotent.** Show that the two definitions of nilpotent given in lecture are the same. Do the same for the two definitions of nilpotency class.

Good Luck!