# Math 741 - Fall 2015

## **Homework 5** Presentations: Mon Nov. 9. 2015 5pm.

1. **Equivalent Notions of Semi-Direct Product.** Let $K, G, Q$ be groups. Show that the following statements are equivalent.

   **a.** There exists a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "\iota"] & G \arrow[r, "pr"] & Q \arrow[l, bend left, "s"] \arrow[r] & 1 \end{tikzcd} \tag{1}$$

   Recall that to say (1) is a short exact sequence means $\iota$ is an injection, $pr$ is a surjection, and $Im(\iota) = Ker(pr)$ (so in particular, $K \trianglelefteq G$). To say the short exact sequence splits means that there exists $s : Q \to G$ such that $pr \circ s = id_Q$.

   **b.** There exists $\tilde{Q} < G$ and $K \trianglelefteq G$ such that $\tilde{Q} \cong Q$, $G = K \cdot \tilde{Q}$ and $K \cap \tilde{Q} = 1$.

   **c.** $G \cong K \rtimes_\alpha Q$ where $\alpha : Q \to Aut(K)$ is a homomorphism. Here $K \rtimes_\alpha Q$ is the group whose underlying set is $K \times Q$ and whose group operation is

$$(k, q) \cdot (k', q') = (k \cdot [\alpha(q)](k'), qq').$$

2. **Not Every Short Exact Sequence Splits.** Let $V = \mathbb{F} \times \mathbb{F}$ for a finite field $\mathbb{F}$ with $char(\mathbb{F}) \neq 2$, and let $w : V \times V \to \mathbb{F}$ be the map defined by $w(u, v) = \det \begin{pmatrix} u \\ v \end{pmatrix}$. Now, let $H = V \times \mathbb{F}$ as a set, and define $\bullet : H \times H \to H$ by

$$(v, z) \bullet (v', z') = (v + v', z + z' + 2^{-1} w(v, v')).$$

   Recall from Homework 4 that $(H, \bullet, (0, 0))$ is a group. Note that we have a short exact sequence

$$0 \to \mathbb{F} \xrightarrow{\iota} H \xrightarrow{pr} V \to 0 \tag{2}$$

   where $\iota, pr$ are the canonical inclusion and projection respectively. Show that (2) does not split. That is, there is no group homomorphism $s : V \to H$ such that $pr \circ s = Id_V$.

3. **Dihedral Group as a Semi-Direct Product.** Consider the set

$$\mathcal{D}_n = \{z \in \mathbb{C} | z^n = 1\} \subseteq \mathbb{C}.$$

Here we consider $\mathbb{C} \cong \mathbb{R}^2$, and we equip $\mathbb{R}^2$ with the standard inner product $<,>$. Let $O(2)$ denote the orthogonal group of $\mathbb{R}^2$ with respect to $<,>$, and consider the group $D_n = Stab_{O(2)}(\mathcal{D}_n)$.

**a.** Show that $D_n$ fits into a short exact sequence

$$1 \to C_n \xrightarrow{\iota} D_n \xrightarrow{d} \{\pm 1\} \to 1.$$

Where $C_n$ is a cyclic group of order $n$.

**b.** Construct a section $s : \{\pm 1\} \to D_n$.

4. **Class Equation.** Write down the class equation for the following groups.

   **a.** $C_n$.

   **b.** $D_n$, from problem 3.

   **c.** $A_4$.

   **d.** $S_4$.

   **e.** $A_5$.

   **f.** $H$, the Heisenberg group from problem 2.

5. **Fixed Point Theorem for Action of p-Groups.** Let $p$ be a prime and $G$ a p-group. That is, $|G| = p^n$ for some $n$. Let $X$ be a finite $G$-set such that $p \nmid |X|$. Show that there exists $x \in X$ such that $gx = x$ for all $g \in G$.

6. **Finite Abelian Groups.** Let $A$ be a finite abelian group and $p$ a prime. Define

$$A_{p'} = \{a \in A |\ p \nmid ord(a)\}.$$

Where $ord(a)$ is the order of $a$. Show that $A_{p'}$ is a subgroup of $A$.

**Good Luck!**