Math 741 - Fall 2015
Finite Heisenberg Group and its Representations

Homework 4 Presentations: Mon Nov. 2. 2015 5pm.

1. **Heisenberg Group.** Let $V = \mathbb{F}_p \times \mathbb{F}_p$ for p an odd prime, and $w : V \times V \to \mathbb{F}_p$ be the bilinear map defined by $w(u,v) = \det \begin{pmatrix} u \\ v \end{pmatrix}$, the determinant of the 2x2 matrix formed by u and v.

   a. Let $H = V \times \mathbb{F}_p$ as a set, and define $\bullet : H \times H \to H$ by
   $$(v,z) \bullet (v',z') = (v+v', z+z'+2^{-1}w(v,v'))$$
   Let $1_H = (0,0)$. Show that $(H, \bullet, 1_H)$ is a group.

   b. Compute $Z(H)$, where $Z(H)$ denotes the center of H.

   c. Compute the number of conjugacy classes of H.

2. **Equivalence of Representations.**

   a. Let G be a group and W a finite dimensional $\mathbb{C}$-vector space. Define what it means that π is a representation of G on W. We denote such a representation by (π, G, W).

   b. Define the notion of equivalence between two representations (π, G, W_π) and (ρ, G, W_ρ)

3. **Irreducible Representations.**

   a. Define what it means for a representation (π, G, W) of a finite group G to be irreducible.

   b. Suppose (π, G, W) is an irreducible representation and $\phi : W \to W$ is an intertwiner. That is, $\phi \circ \pi(g) = \pi(g) \circ \phi$ for all $g \in G$. Show that there exists $\alpha \in \mathbb{C}$ such that $\phi = \alpha \cdot Id_W$.

4. **One Dimensional Representations of H.** Recall the definitions of V and H from problem 1.

a. Write down an explicit formula for all 1-dimensional representations of V.

b. Using the canonical projection $\pi : H \to V$, write down p^2 one-dimensional representations of H.

5. **Heisenberg Representation.** Let $\psi : \mathbb{F}_p \to \mathbb{C}^*$ be an additive character. That is, a map such that $\psi(x+y) = \psi(x)\psi(y)$. Further, suppose $\psi \neq 1$, and let $\mathcal{H} = \mathbb{C}(\mathbb{F}_p)$ be the $\mathbb{C}$-vector space of complex valued functions on $\mathbb{F}_p$. Define $\pi_\psi : H \to GL(\mathcal{H})$ by

$$[\pi_\psi(x,y,z)(f)](t) = \psi(2^{-1}xy + z)\psi(yt)f(t+x)$$

for all $f \in \mathcal{H}$. Prove that π_ψ is a homomorphism (that is, a representation of H on $\mathcal{H}$).

6. **Representations of H.**

a. Show that for each ψ as in problem 5, the representation π_ψ is irreducible.

b. Show that if $\psi \neq \psi'$, then the representations π_ψ and $\pi_{\psi'}$ are not isomorphic.

c. Let $Irr(H)$ denote the set of irreducible representations of H modulo equivalence. Show that problems 4 and 5 give representatives for every equivalence class in $Irr(H)$. (You may use the fact that for a finite group G, the order of $Irr(G)$ is equal to the number of conjugacy classes of G).

Good Luck!