Math 741 - Fall 2015

Homework 3 Presentations: Mon Oct 19. 2015 5pm.

1. **Transitive Group Actions.** We say an action of a group G on a set X is transitive if, for any $x, y \in X$, there exists some $g \in G$ such that $g \bullet x = y$. Suppose G acts transitively on X.

   i. Let G_x be the stabilizer subgroup of x in G. Construct a bijection $i_x : G/G_x \to X$.

   ii. Given any $x, y \in X$, show that there exists $g \in G$ such that $gG_xg^{-1} = G_y$.

2. **Bilinear Forms.** Let $V = \mathbb{F}_q \times \mathbb{F}_q$ for q odd and consider the bilinear form $B : V \times V \to \mathbb{F}_q$ given by

$$B((x,y),(x',y')) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

   Recall that B' is equivalent to B if there exists $g \in GL_2(\mathbb{F}_q)$ such that $B'(gu, gv) = B(u,v)$ for all $u, v \in V$. Compute the number of bilinear forms on V which are equivalent to B.

3. **Normal Subgroups.** Let H and N be subgroups of G, with N being normal.

   i. Show that $N \cap H$ is a normal subgroup of H.

   ii. Show that HN is a subgroup of G.

   iii. Show that $HN/N \cong H/(H \cap N)$.

4. **Intermediate Subgroups.** Let H and N be subgroups of G, with N normal and $N < H < G$.

   i. Show that H/N is a subgroup of G/N.

   ii. Now assume that H is normal in G. Show that H/N is normal in G/N.

   iii. Let $pr : G/N \to G/H$ be the natural projection. Show that $ker(pr) = H/N$.

   iv. Explain why $(G/N)/(H/N) \cong G/H$.

5. **Product as Terminal Object.** Let $\mathcal{C}$ be a category, with $A, B \in Ob(\mathcal{C})$. We define a new category $\mathcal{C}_{A,B}$ as follows. The objects of $\mathcal{C}_{A,B}$ are triples (X, π_A, π_B) such that X is an object in $\mathcal{C}$, and $\pi_A : X \to A$, $\pi_B : X \to B$ are morphisms in $\mathcal{C}$. A morphism in $\mathcal{C}_{A,B}$ from an object (X, π_A, π_B) to an object (Y, π'_A, π'_B) is a morphism $f : X \to Y$ in $\mathcal{C}$ such that everything commutes. Namely, $\pi'_A \circ f = \pi_A$ and similarly $\pi'_B \circ f = \pi_B$.

   Recall that an object T in a category is said to be <u>terminal</u> if for any other object X in the category, there exists a unique morphism $\tau_X : X \to T$. In the cases $\mathcal{C} = Set, Grp, Vect$, show that $\mathcal{C}_{A,B}$ has a terminal object, for any choice of A, B.

6. **Stabilizers.** Let H be a subgroup of G. Show that there exists a set X, along with an action of G on X, and a point $x \in X$ such that $Stab_G(x) = H$.

Good Luck!